Synchronisation of coupled oscillators: From Huygens clocks to chaotic systems and large ensembles

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What is synchronization

σύν: *syn* = the same, common   χρόνος: *chronos* = time

Synchronization: Adjustment of rhythms of oscillating objects due to an interaction

Periodic oscillators

Synchronization by external force

Mutual synchronization of two oscillators

Synchronization in oscillatory media

Populations of coupled oscillators

Synchronization by common noise

Chaotic oscillators

Complete/identical synchronization

Phase synchronization

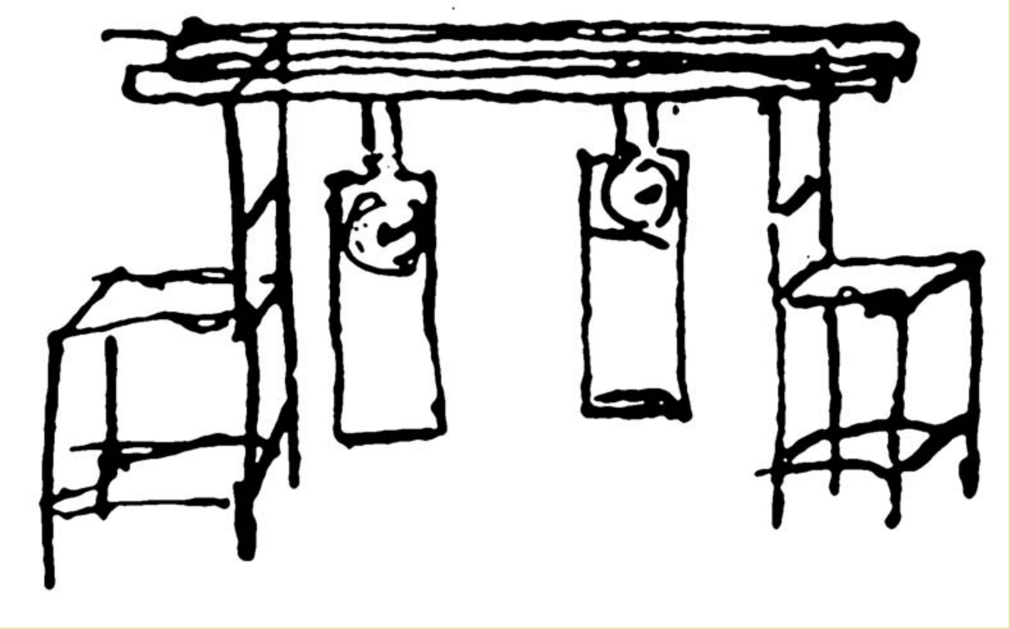
Generalized, master-slave, replica, . . .

Historical introduction

Christiaan Huygens (1629-1695) first observed a synchronization of two pendulum clocks

He described:

"... It is quite worths noting that when we suspended two clocks so constructed from two hooks imbedded in the same wooden beam, the motions of each pendulum in opposite swings were so much in agreement that they never receded the least bit from each other and the sound of each was always heard simultaneously. Further, if this agreement was disturbed by some interference, it reestablished itself in a short time. For a long time I was amazed at this unexpected result, but after a careful examination finally found that the cause of this is due to the motion of the beam, even though this is hardly perceptible."

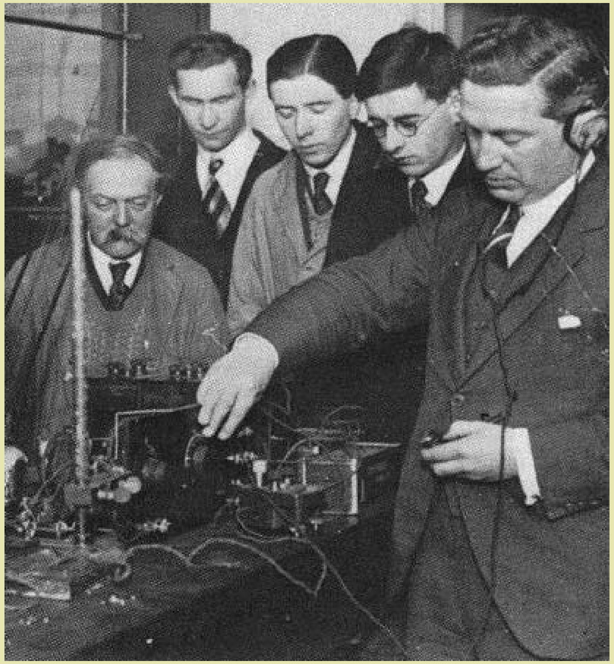


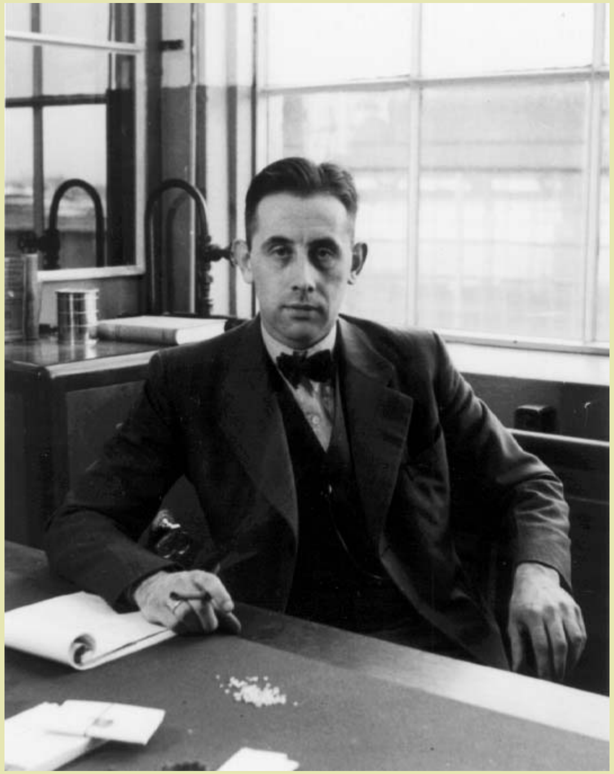
Lord Rayleigh described synchronization in acoustical systems:

"When two organ-pipes of the same pitch stand side by side, complications ensue which not unfrequently give trouble in practice. In extreme cases the pipes may almost reduce one another to silence. Even when the mutual influence is more moderate, it may still go so far as to cause the pipes to speak in absolute unison, in spite of inevitable small differences."

W. H. Eccles and J. H. Vincent applied for a British Patent confirming their discovery of the synchronization property of a triode generator

Edward Appleton and **Balthasar van der Pol** extended the experiments of Eccles and Vincent and made the first step in the theoretical study of this effect (1922-1927)





Jean-Jacques Dortous de Mairan reported in 1729 on his experiments with the haricot bean and found a circadian rhythm (24-hours-rhythm): motion of leaves continues even without variations of the illuminance

Engelbert Kaempfer wrote after his voyage to Siam in 1680:

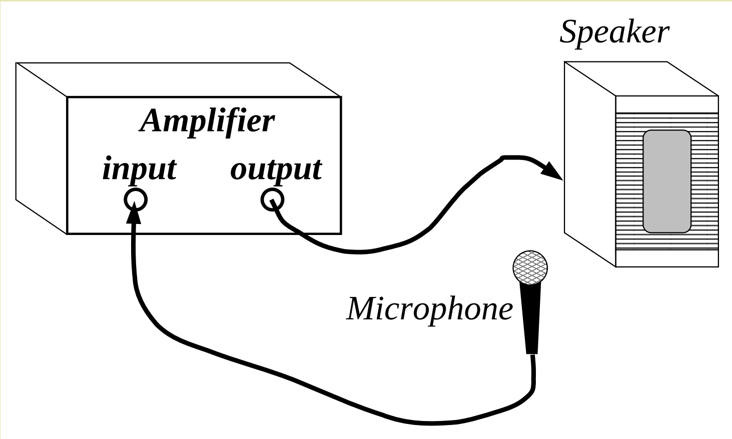
> "The glowworms . . . represent another shew, which settle on some Trees, like a fiery cloud, with this surprising circumstance, that a whole swarm of these insects, having taken possession of one Tree, and spread themselves over its branches, sometimes hide their Light all at once, and a moment after make it appear again with the utmost regularity and exactness . . .".

Self-sustained oscillators

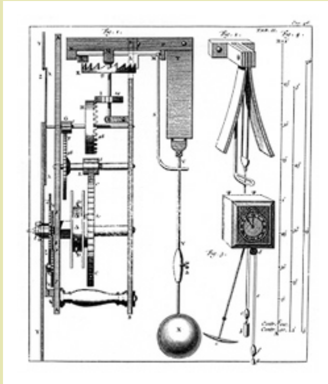
- generate periodic oscillations without periodic forces
- are dissipative nonlinear systems described by autonomous ODEs
- possess a limit cycle in the phase space

Laser, periodic chemical reactions, predator-pray system, violine, ...

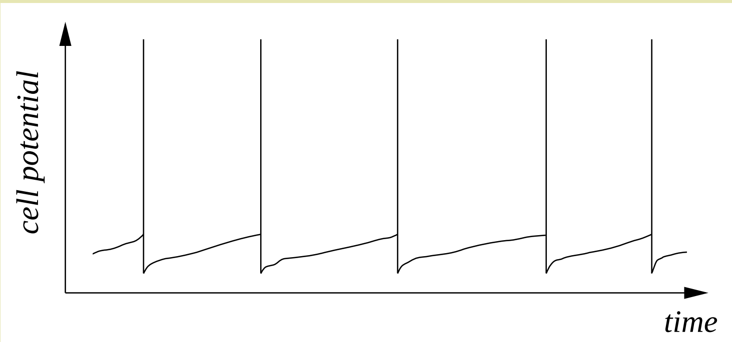
positive feedback loop



pendulum clock



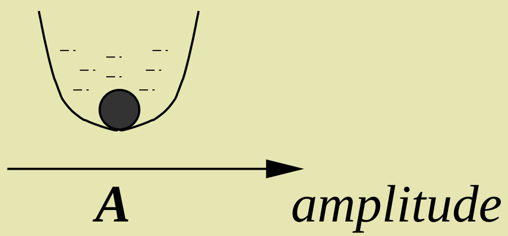
firing neuron

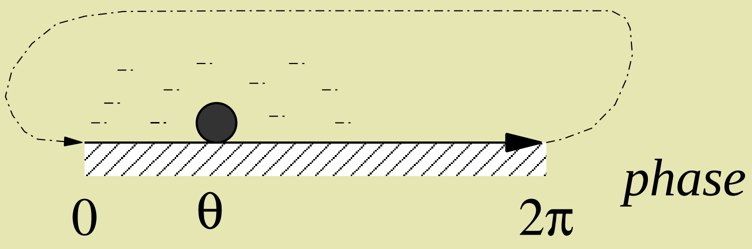


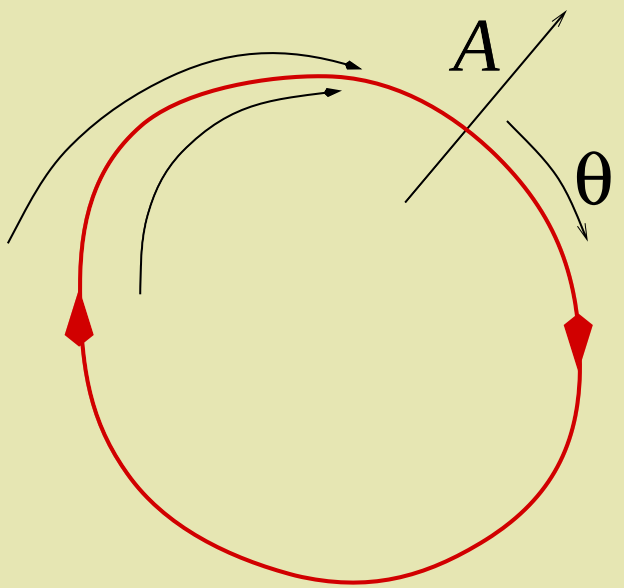
metronom

Autonomous oscillator

- amplitude (form) of oscillations is fixed and stable
- **PHASE** of oscillations is free due to the time-shift invariance







$$\dot{\theta} = \omega_0$$ (Lyapunov exp. 0)

$$\dot{A} = -\gamma(A - A_0)$$ (Lyapunov exp. $-\gamma$)

Forced oscillator

With small periodic external force (e.g. $\sim \varepsilon \sin \omega t$):

only the phase θ is affected

$$\frac{d\theta}{dt} = \omega_0 + \varepsilon G(\theta, \psi) \qquad \frac{d\psi}{dt} = \omega$$

ψ is the phase of the external force, $G(\cdot,\cdot)$ is 2π-periodic

If $\omega_0 \approx \omega$ then the phase difference $\varphi = \theta(t) - \psi(t)$ is slow

$\Rightarrow$ perform averaging by keeping only slow terms (e.g. $\sim \sin(\theta - \psi)$)

$$\boxed{\frac{d\varphi}{dt} = \Delta\omega + \varepsilon \sin \varphi}$$

Parameters in the **Adler equation**:

$\Delta\omega = \omega_0 - \omega$ detuning

ε forcing strength

Solutions of the Adler equation

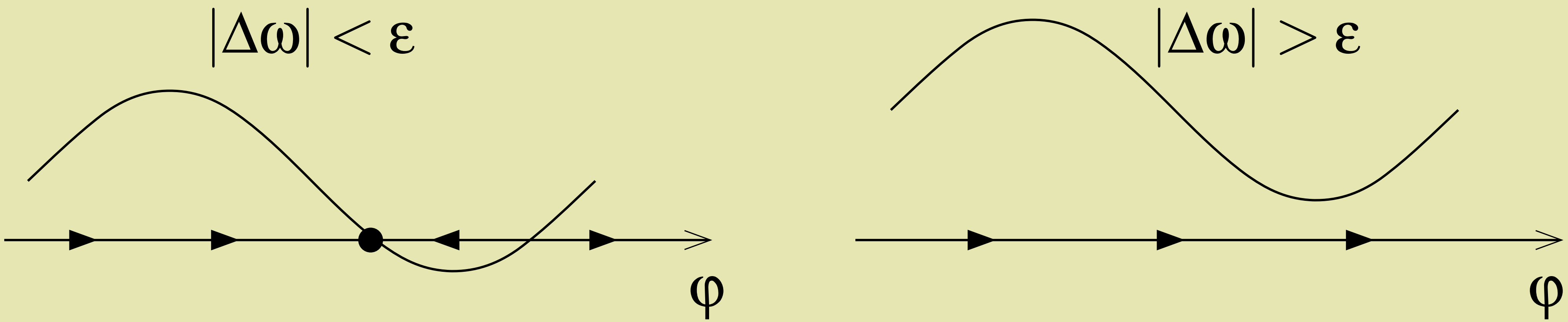
$$\boxed{\frac{d\varphi}{dt} = \Delta\omega + \varepsilon \sin\varphi}$$

Fixed point for $|\Delta\omega| < \varepsilon$:

Frequency entrainment $\Omega = \langle \dot{\theta} \rangle = \omega$

Phase locking $\varphi = \theta - \psi =$ const

Periodic orbit for $|\Delta\omega| > \varepsilon$: an asynchronous quasiperiodic motion

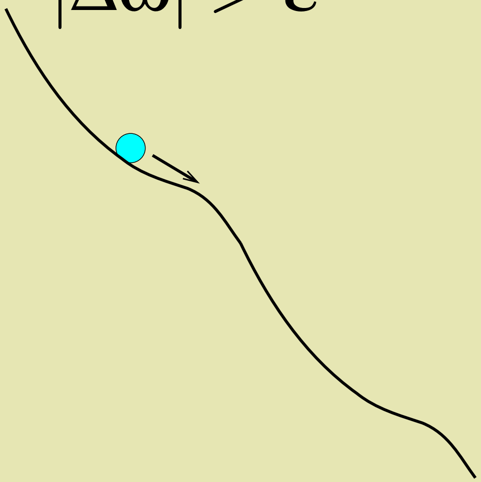


Phase dynamics as a motion of an overdamped particle in an inclined potential

$$\frac{d\varphi}{dt} = -\frac{dU(\varphi)}{d\varphi} \qquad U(\varphi) = -\Delta\omega \cdot \varphi + \varepsilon \cos\varphi$$

$|\Delta\omega| < \varepsilon$

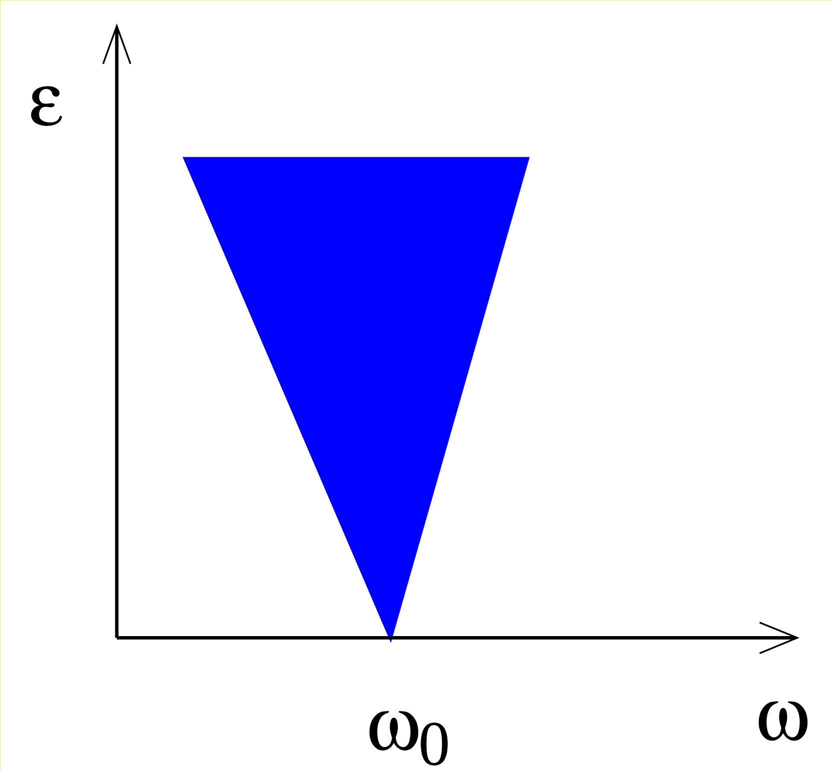
$|\Delta\omega| > \varepsilon$

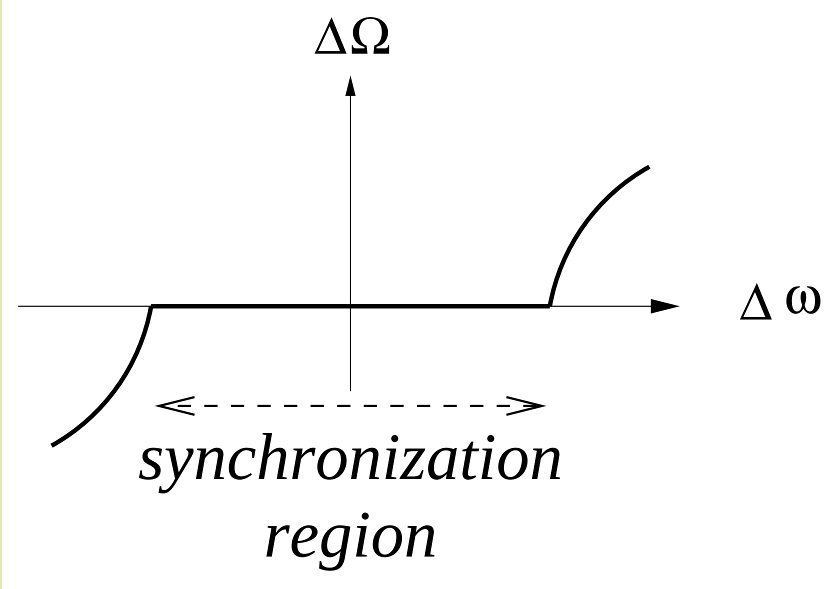


Allows one to understand what happens in presence of noise:

no perfect locking but **phase slips** due to excitation over the barrier

Synchronization region = Arnold tongue





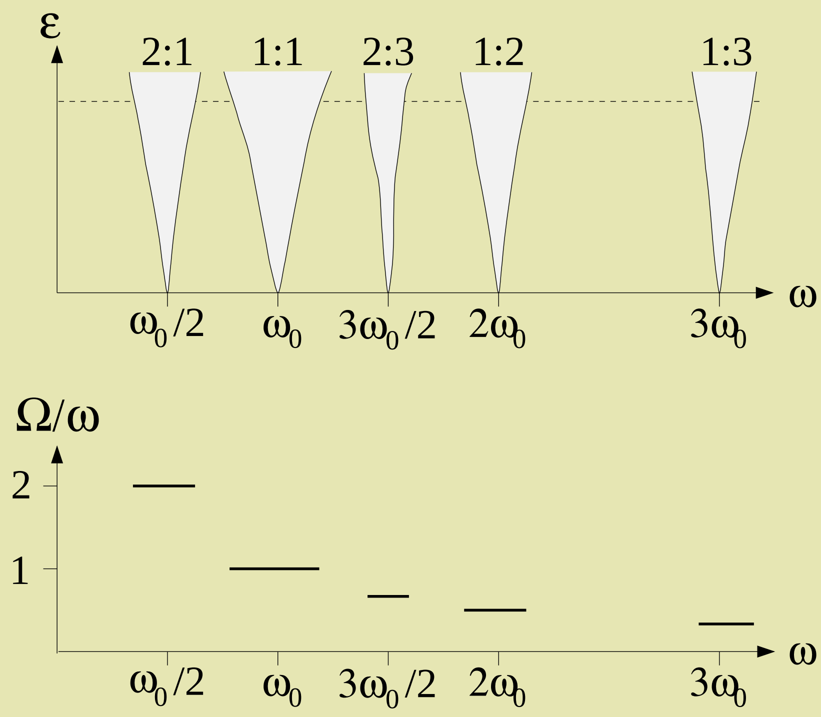
Unusual situation: synchronization occurs for very small force $\varepsilon \longrightarrow 0$,

but cannot be obtain with a (linear) perturbation method:

the perturbation theory is singular due to a degeneracy

(vanishing Lyapunov exponent)

More generally: synchronization of higher order is possible, whith a relation $\frac{\Omega}{\omega} = \frac{m}{n}$



Mathematical description: reduce the system on a torus

$$\frac{d\phi}{dt} = \omega_0 + \varepsilon G(\phi, \psi) \qquad \frac{d\psi}{dt} = \nu$$

to a **circle map**

$$\phi_{n+1} = \phi_n + \nu + \varepsilon g(\phi_n)$$

$$\phi_{n+1} = \phi_n + \nu + \varepsilon g(\phi_n)$$

Rotation number = average number of rotations pro iteration

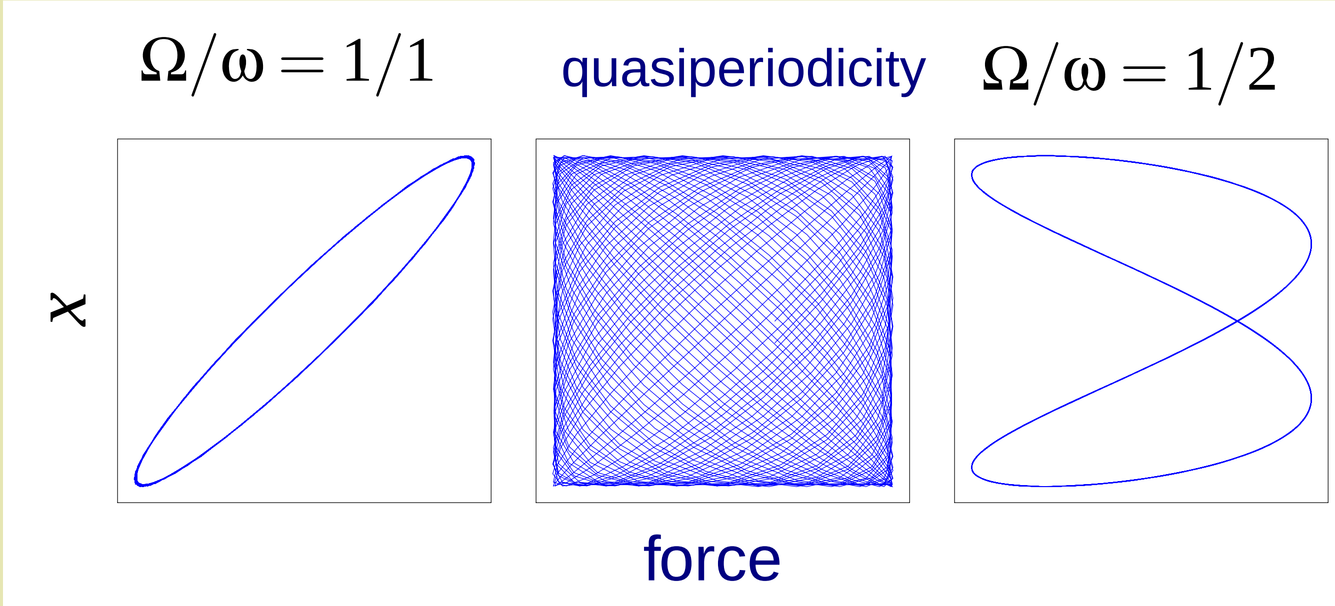
$$\rho = \lim_{T \to \infty} \frac{1}{2\pi} \frac{\phi_T - \phi_0}{T}$$

- $\rho = \frac{p}{q}$ rational: stable and unstable periodic orbits
- ρ irrational: quasiperiodic dense filling of the circle

For the continuous-time system: $\rho = \frac{\langle \dot{\phi} \rangle}{\langle \dot{\psi} \rangle}$ = ratio of the frequencies

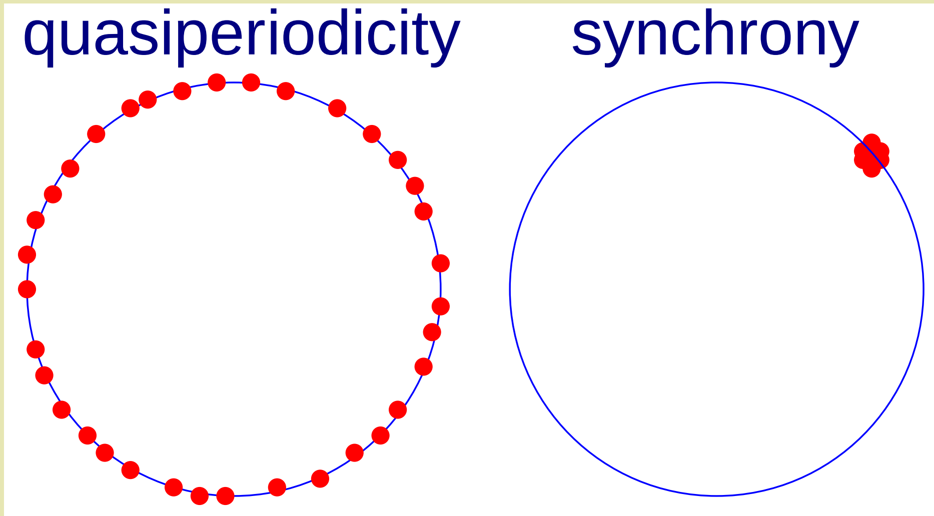
The simplest ways to observe synchronization:

Lissajous figure

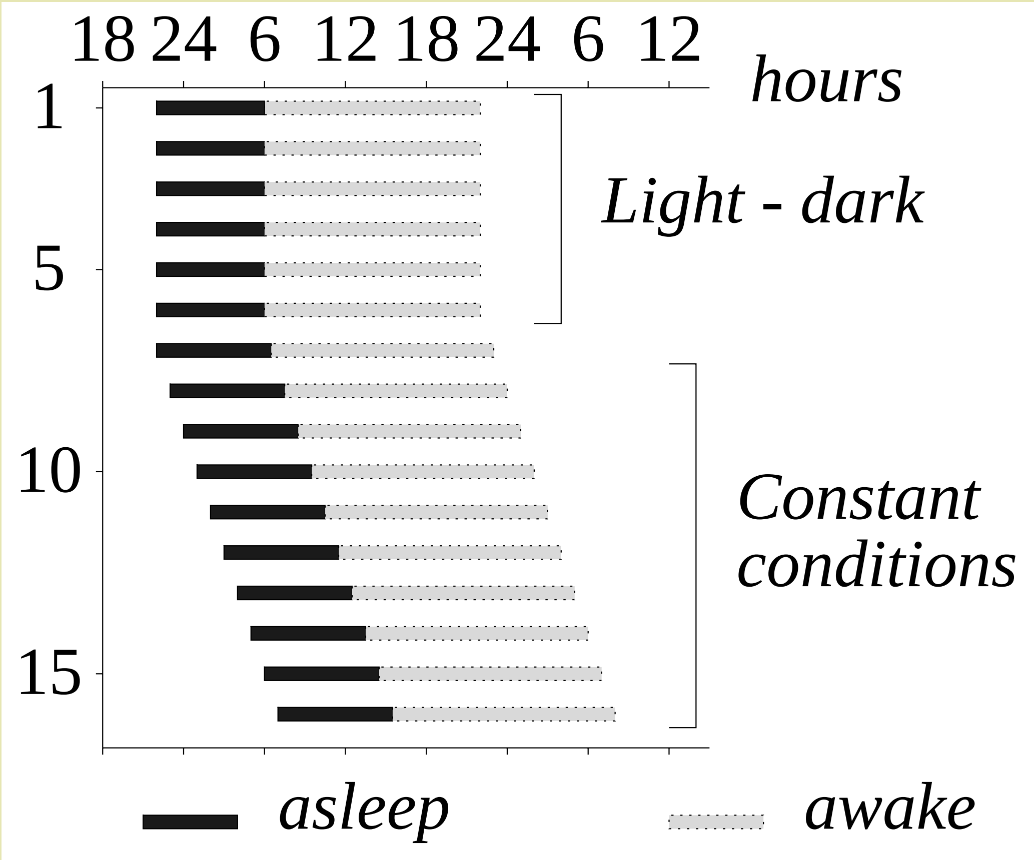


Stroboscopic observation:

Plot phase at each period of forcing



Example: circadian rhythm



The “Jet-Lag” results from the phase shift of the force – a new entrainment takes some time

Example: radio-controlled clocks

Atomic clocks at the German institute of standards (PTB) in Braunschweig

radio-controlled clock

$\Rightarrow$

Mutual synchronization

Two non-coupled self-sustained oscillators:

$$\frac{d\theta_1}{dt} = \omega_1 \qquad \frac{d\theta_2}{dt} = \omega_2$$

Two weakly coupled oscillators:

$$\frac{d\theta_1}{dt} = \omega_1 + \varepsilon G_1(\theta_1, \theta_2) \qquad \frac{d\theta_2}{dt} = \omega_2 + \varepsilon G_2(\theta_1, \theta_2)$$

For $\omega_1 \approx \omega_2$ the phase difference $\varphi = \theta_1 - \theta_2$ is slow

$\Rightarrow$ averaging leads to the **Adler equation**

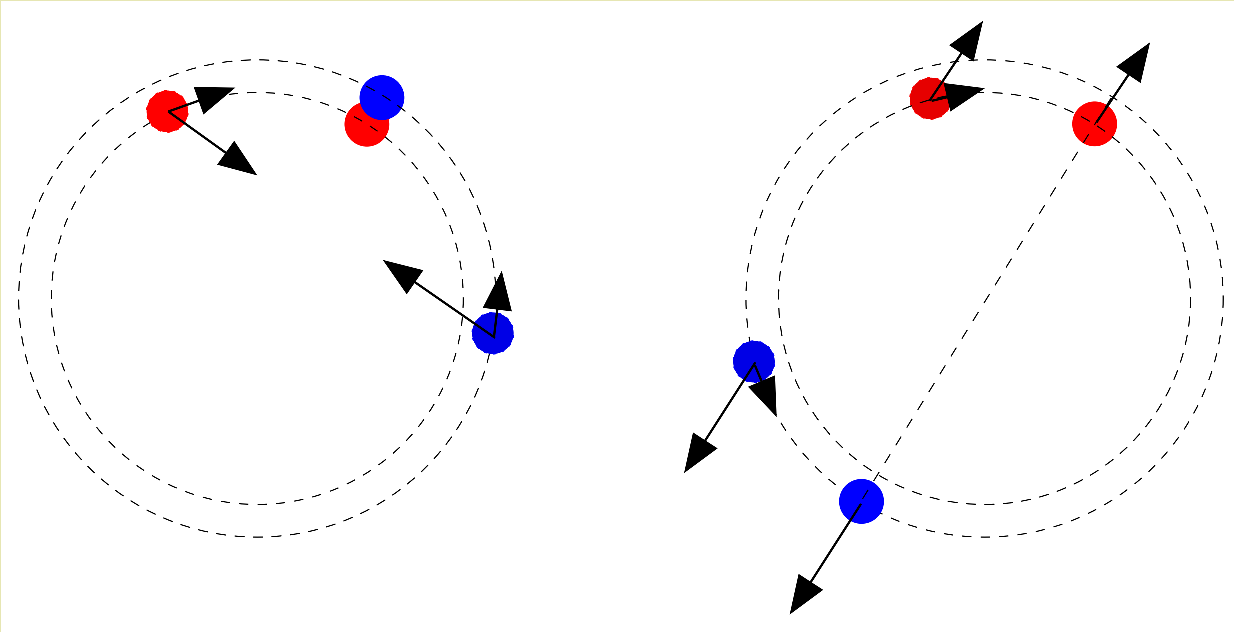
$$\boxed{\frac{d\varphi}{dt} = \Delta\omega + \varepsilon \sin\varphi}$$

Parameters:

$\Delta\omega = \omega_1 - \omega_2$ detuning

ε coupling strength

Interaction of two periodic oscillators may be attractive ore repulsive: one observes **in phase** or **out of phase** synchronization, correspondingly



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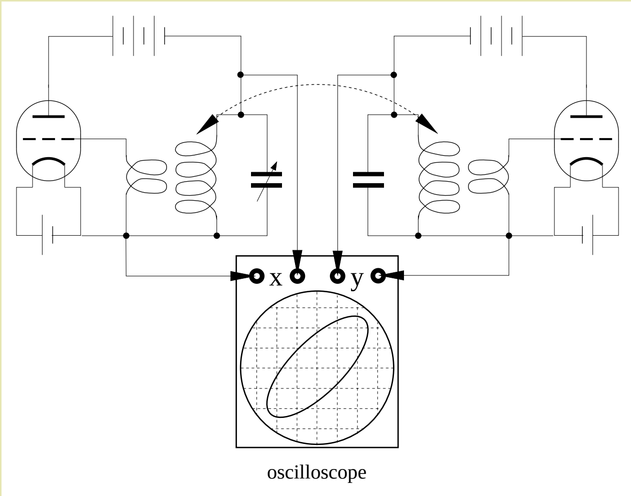
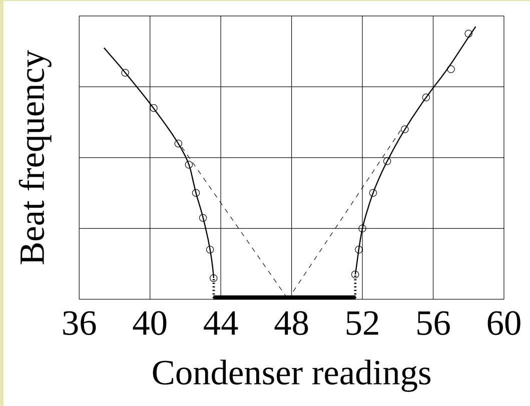
Example: classical experiments by Appleton





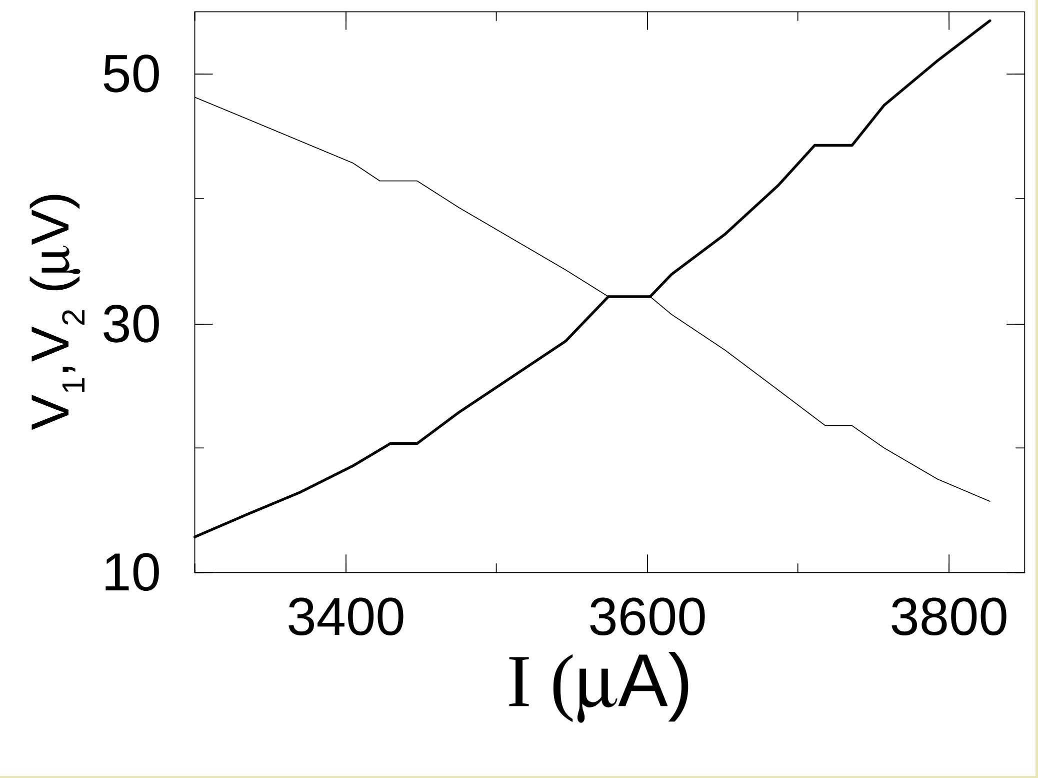
Huygens pendulum clocks: see Bennett, Schatz, Rockwood and Wiesenfeld, Proc. R. Soc. Lond. A (2002)

Organ pipes: see Abel, Bergweiler and Gerhard-Multhaupt, J. Acoust. Soc. Am. (2006)

Synchronization of Josephson junctions

Josephson junction (= pendulum) is a rotator, has a zero Lyapunov exponent

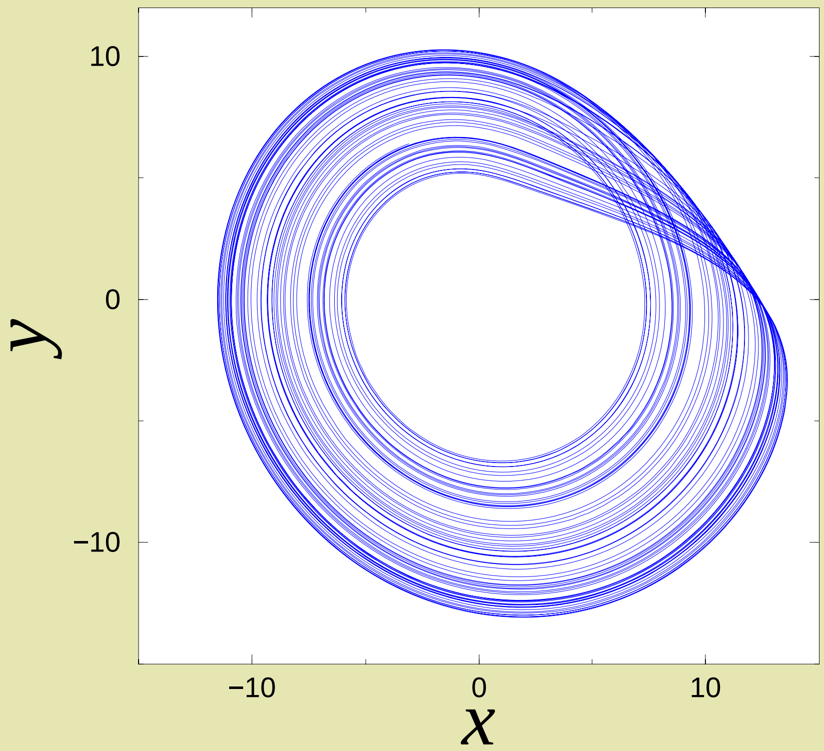
Voltage $= \frac{\hbar}{2e}\langle\dot{\theta}\rangle$ measures the frequency



Phase of a chaotic oscillator

Rössler attractor:

$$\begin{aligned} \dot{x} &= -y - z \\ \dot{y} &= x + 0.15y \\ \dot{z} &= 0.4 + z(x - 8.5) \end{aligned}$$

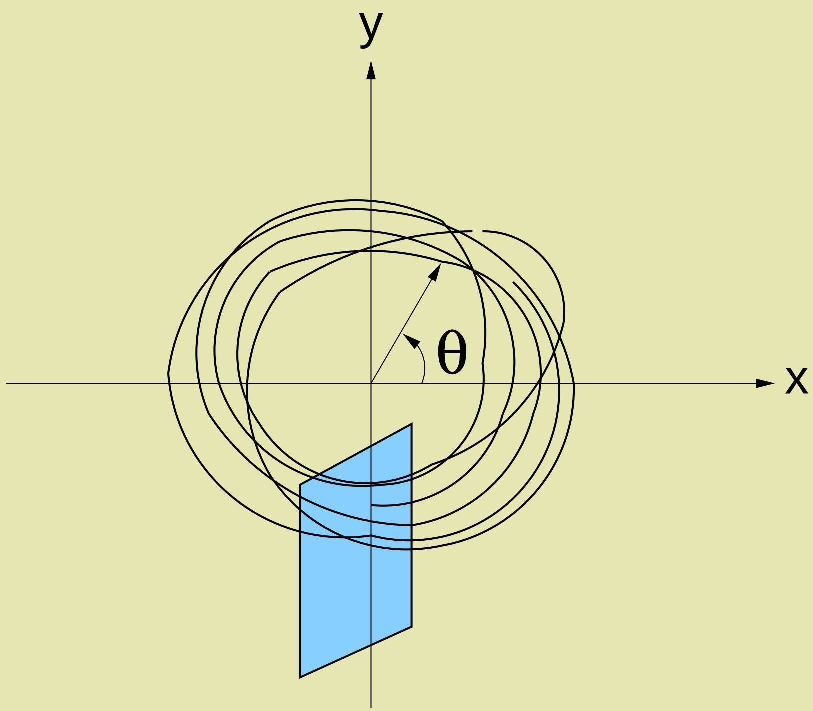


phase should correspond to the zero Lyapunov exponent!

naive definition of the phase: $\theta = \arctan(y/x)$

basing on the Poincaré map:

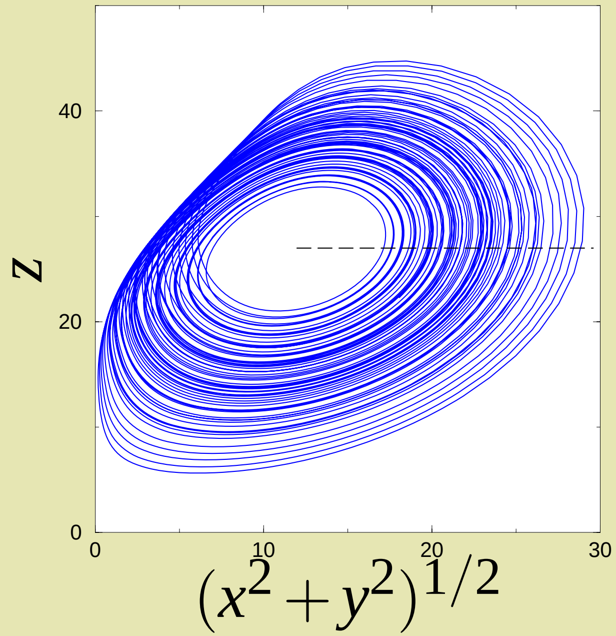
$$\theta = 2\pi \frac{t - t_n}{t_{n+1} - t_n} + 2\pi n \qquad t_{n+1} \leq t < t_n$$

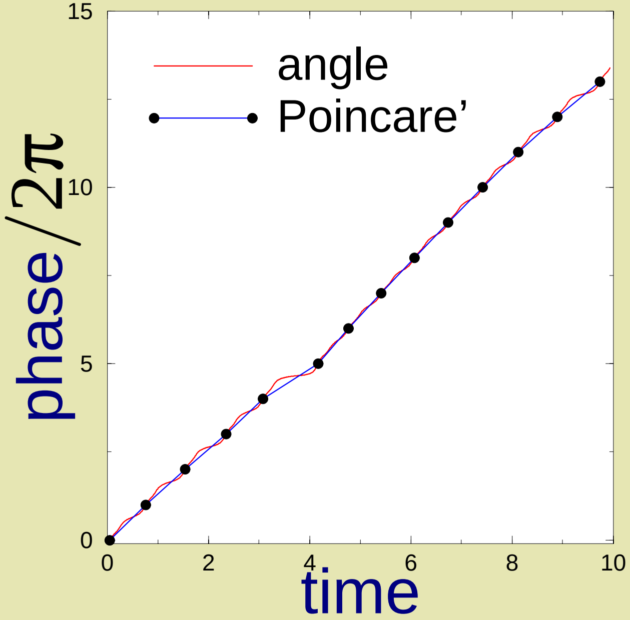


For the topologically simple attractors all definitions are good

Lorenz attractor:

$$\dot{x} = 10(y - x)$$
$$\dot{y} = 28x - y - xz$$
$$\dot{z} = -8/3z + xy$$





Phase dynamics in a chaotic oscillator

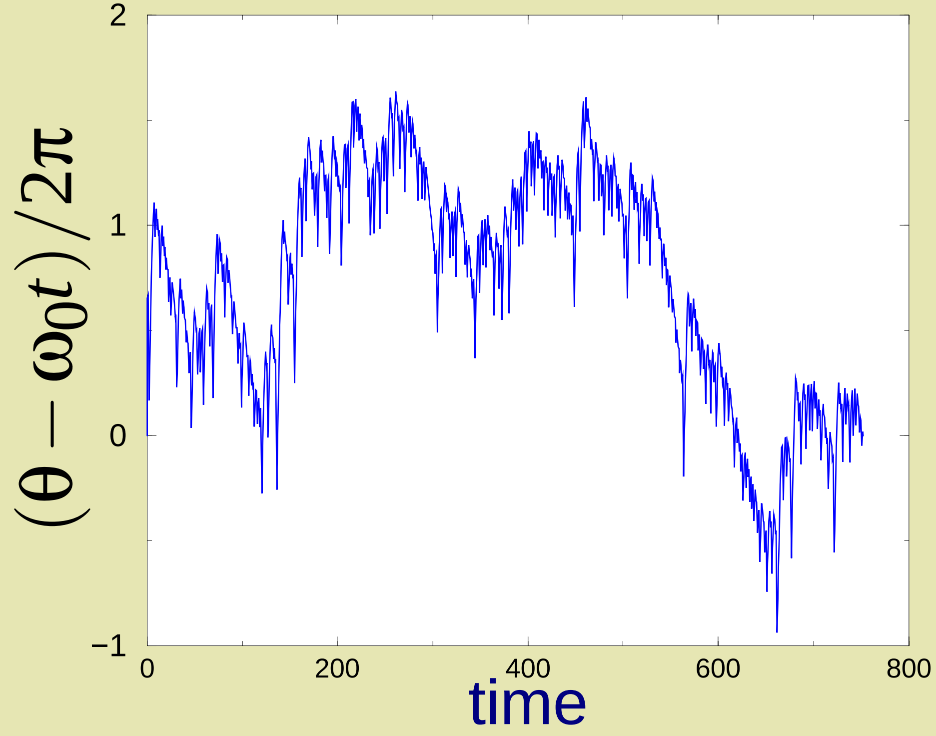
A model phase equation: $\frac{d\theta}{dt} = \omega_0 + F(A)$

(first return time to the surface of section depends on the coordinate on the surface)

A: chaotic $\Rightarrow$ phase diffusion $\Rightarrow$ broad spectrum

$$\langle(\theta(t) - \theta(0) - \omega_0 t)^2\rangle \propto D_p t$$

D_p **measures coherence of chaos**



$$\frac{d\theta}{dt} = \omega_0 + F(A)$$

$F(A)$ is like effective noise $\Rightarrow$

Synchronization of chaotic oscillators $\approx$

$\approx$ synchronization of noisy periodic oscillators $\Rightarrow$

phase synchronization can be observed while the “amplitudes” remain chaotic

Synchronization of a chaotic oscillator by external force

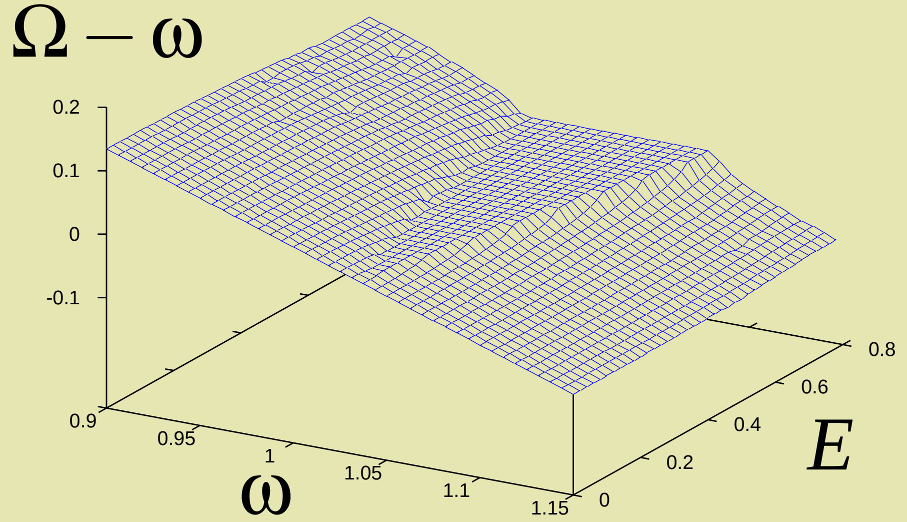
If the phase is well-defined $\Rightarrow \Omega = \langle \frac{d\theta}{dt} \rangle$ is easy to calculate

(e.g. $\Omega = 2\pi \lim_{t\to\infty} N_t/t$, N is a number of maxima)

Forced

Rössler oscillator:

$$
\begin{aligned}
\dot{x} &= -y - z + E\cos(\omega t) \\
\dot{y} &= x + ay \\
\dot{z} &= 0.4 + z(x - 8.5)
\end{aligned}
$$

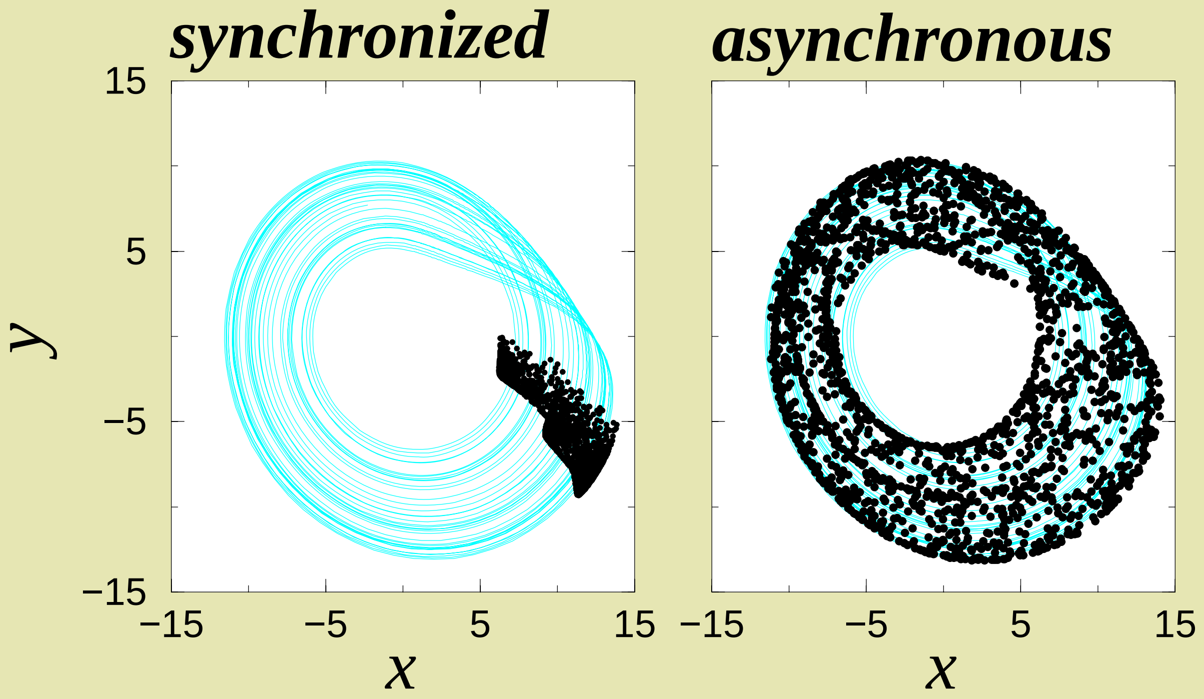


phase is locked, amplitude is chaotic

Stroboscopic observation

Autonomous chaotic oscillator: phases are distributed from 0 to 2π.

Under periodic forcing: if the phase is locked, then the distribution has a sharp peak near $\theta = \omega t + const$.



Phase synchronization of chaotic gas discharge by periodic pacing

Tacos et al, Phys. Rev. Lett. **85**, 2929 (2000)

Experimental setup:

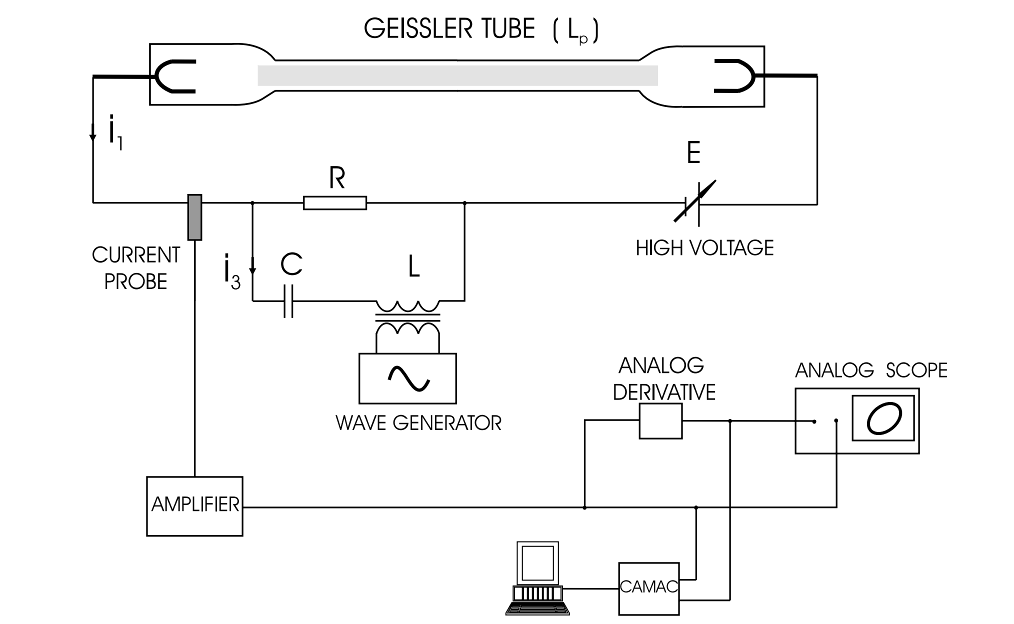
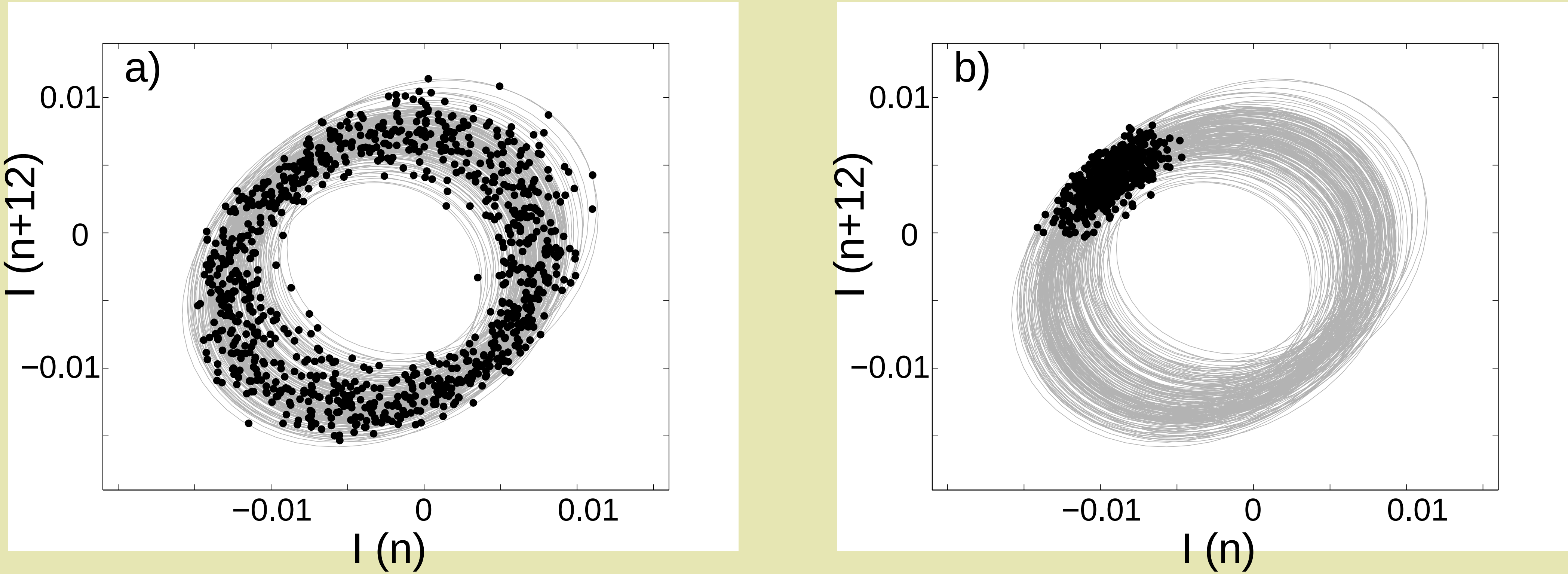
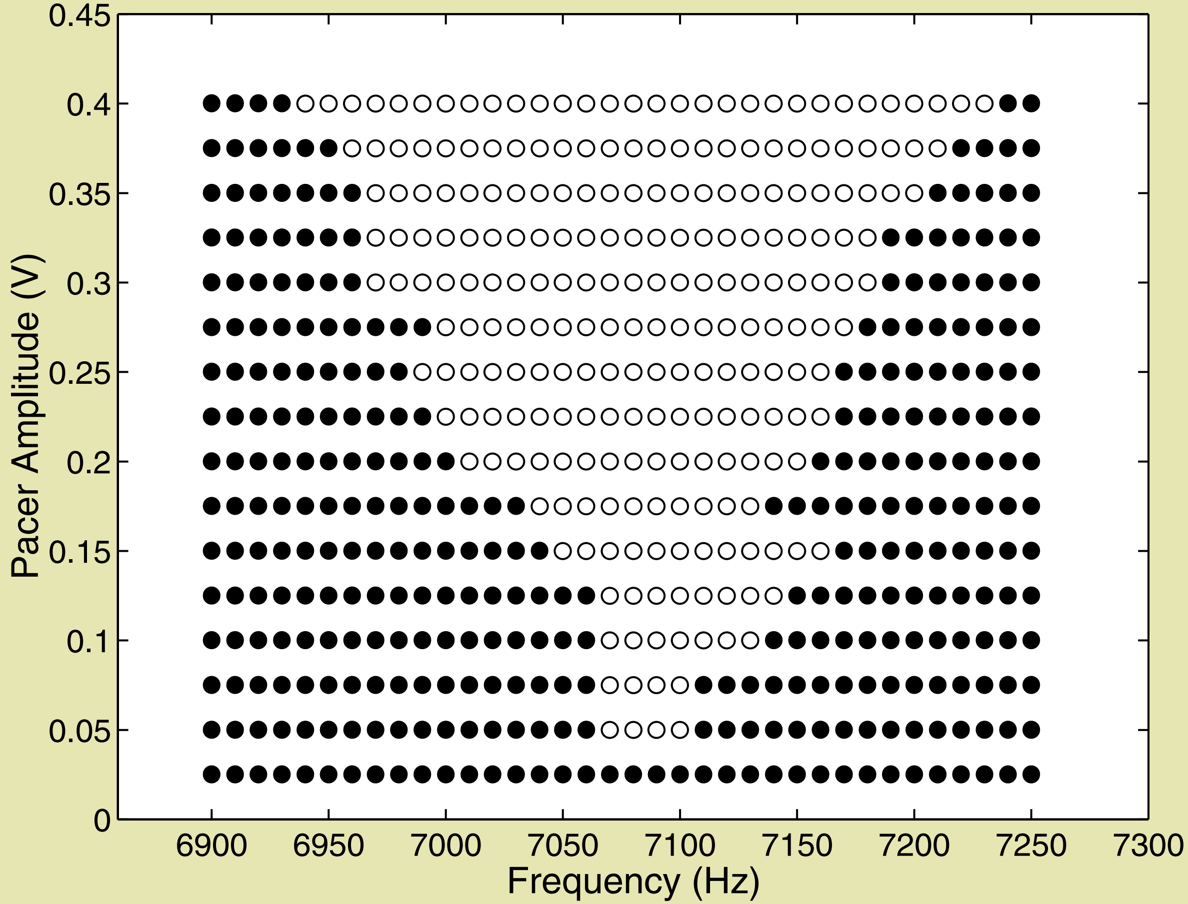


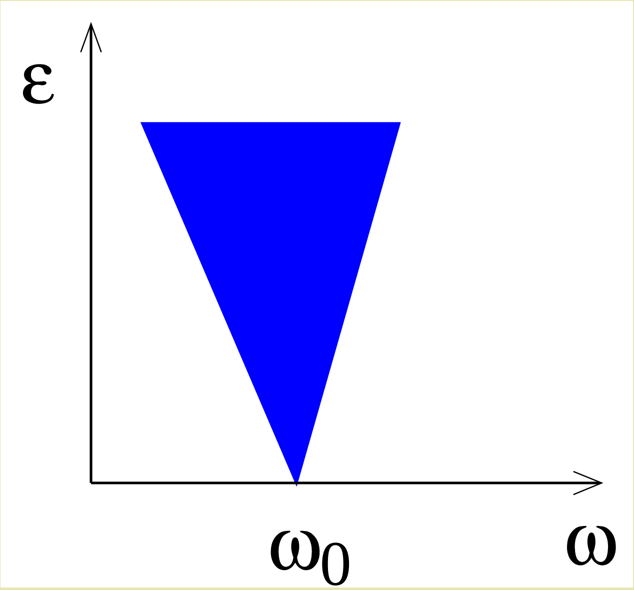
FIG. 2. Schematic representation of our experimental setup.

Phase plane projections in non-synchronized and synchronized cases



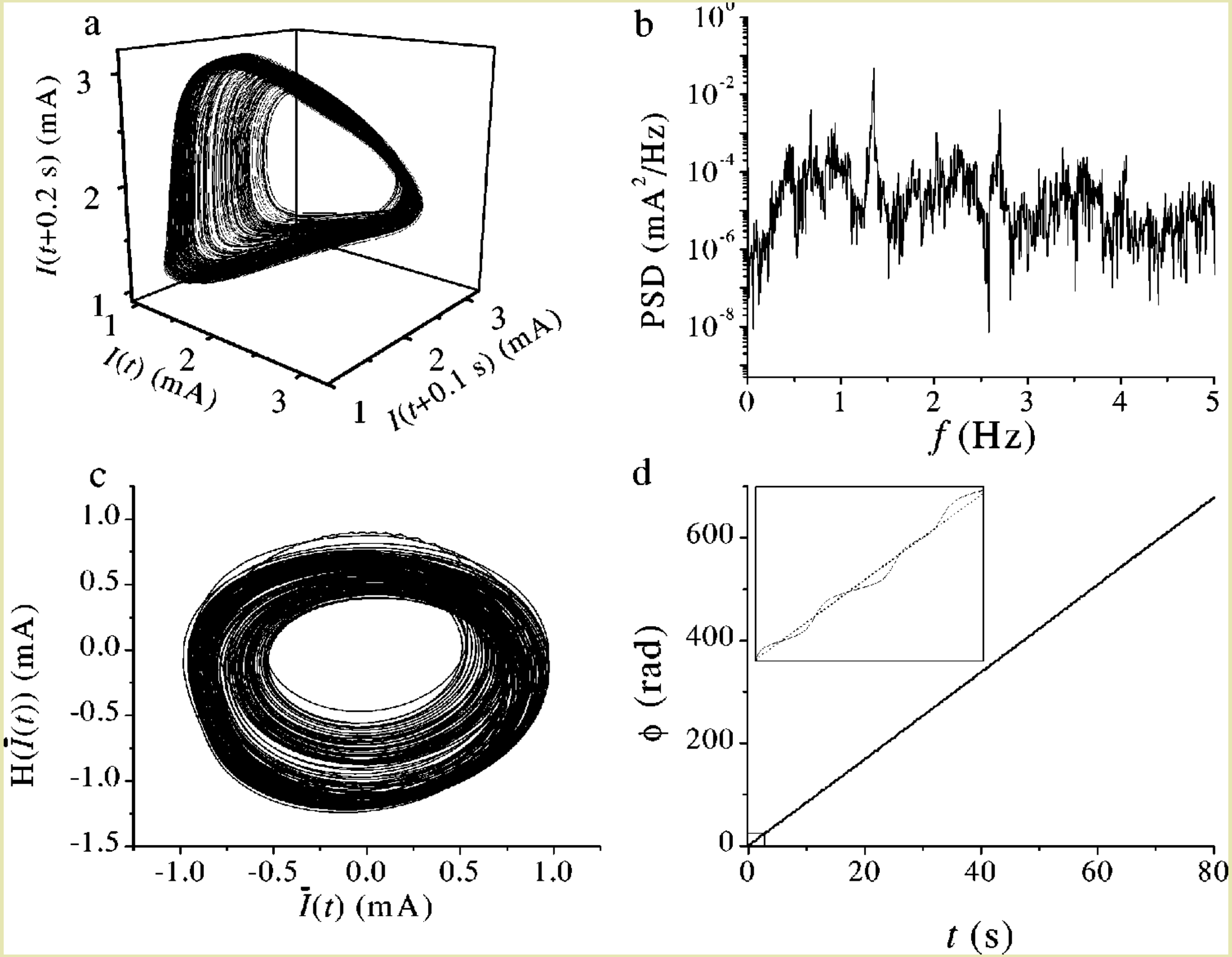
Synchronization region:



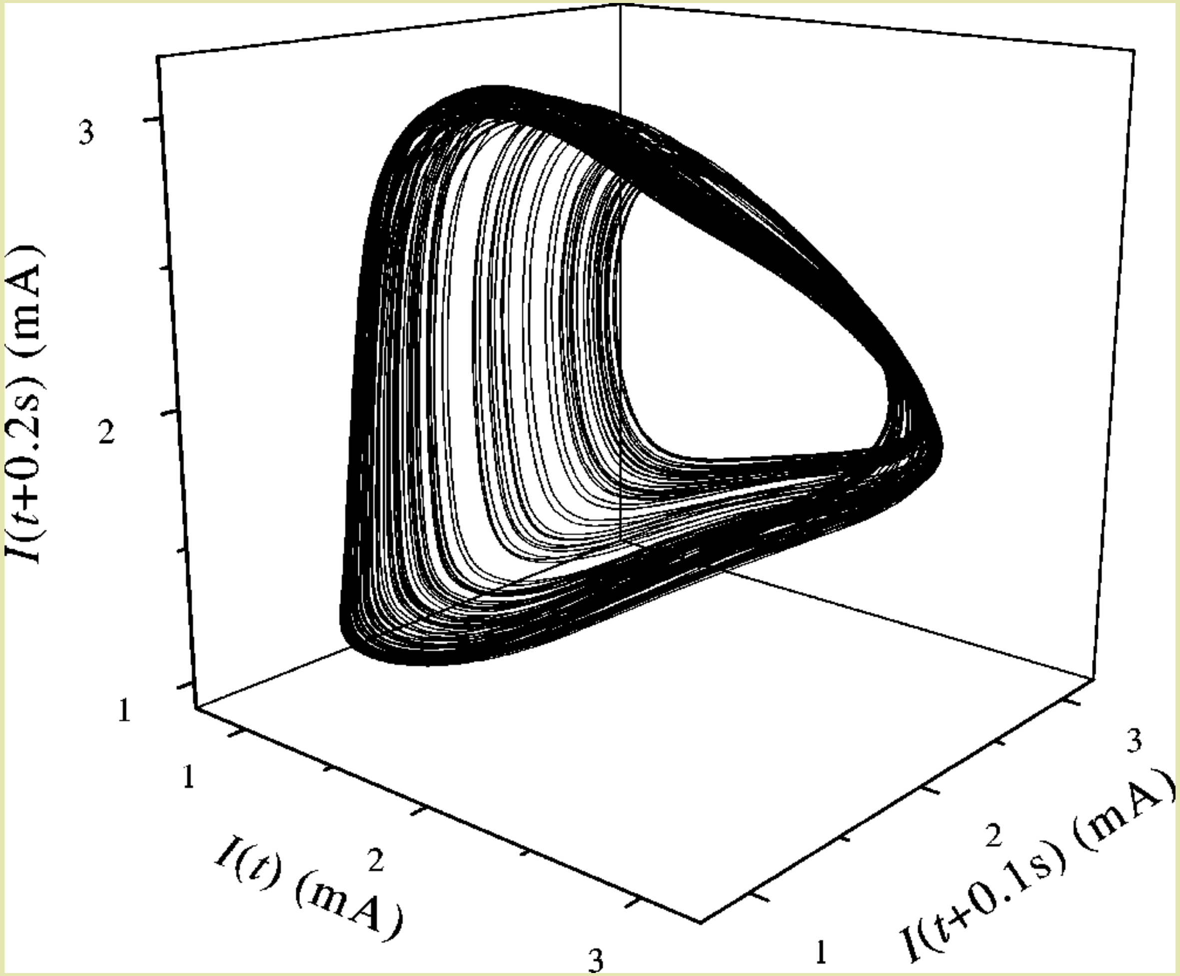


Electrochemical chaotic oscillator

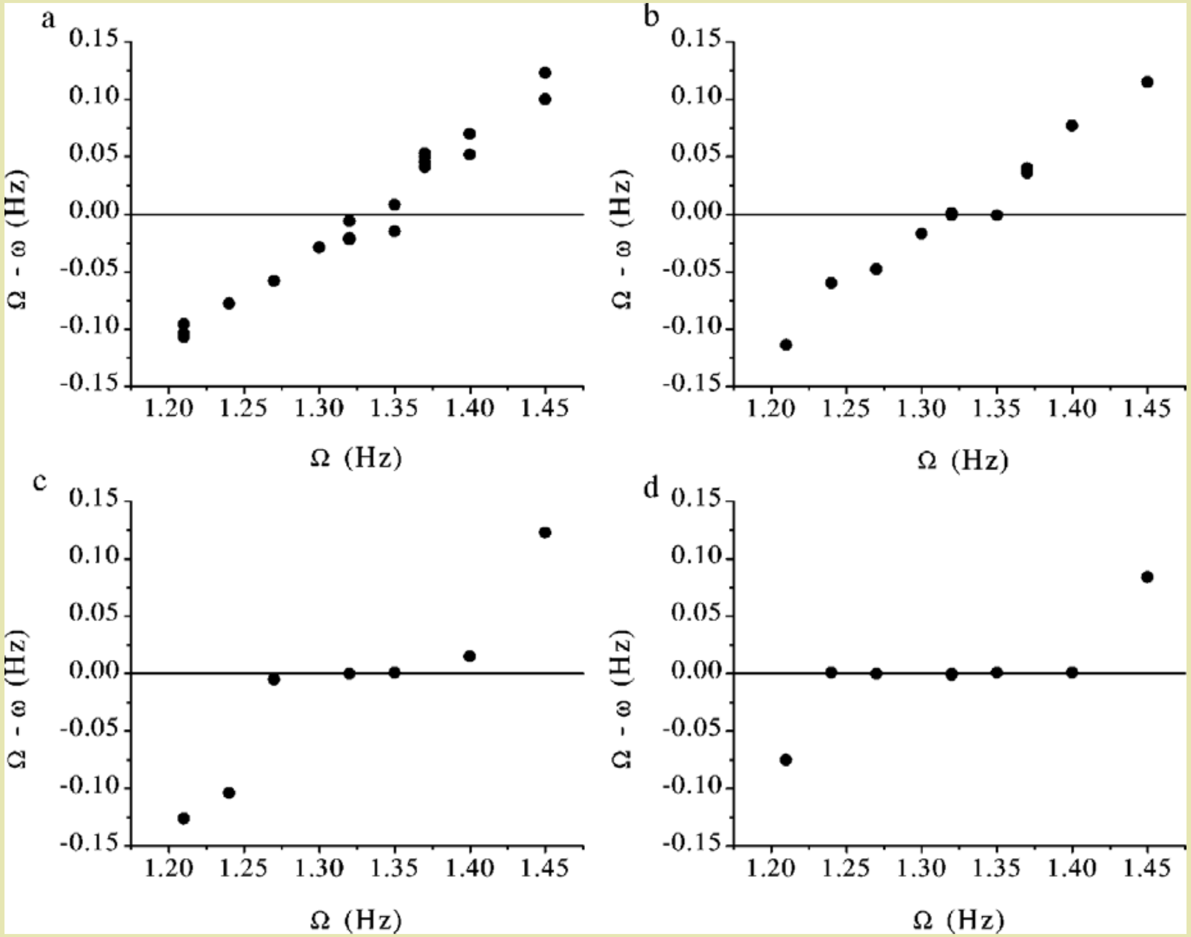
Kiss and Hudson, Phys. Rev. E **64**, 046215 (2001)

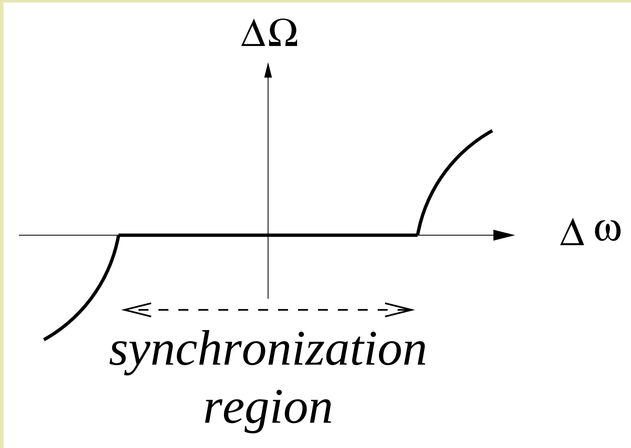


The synchronized oscillator remains chaotic:

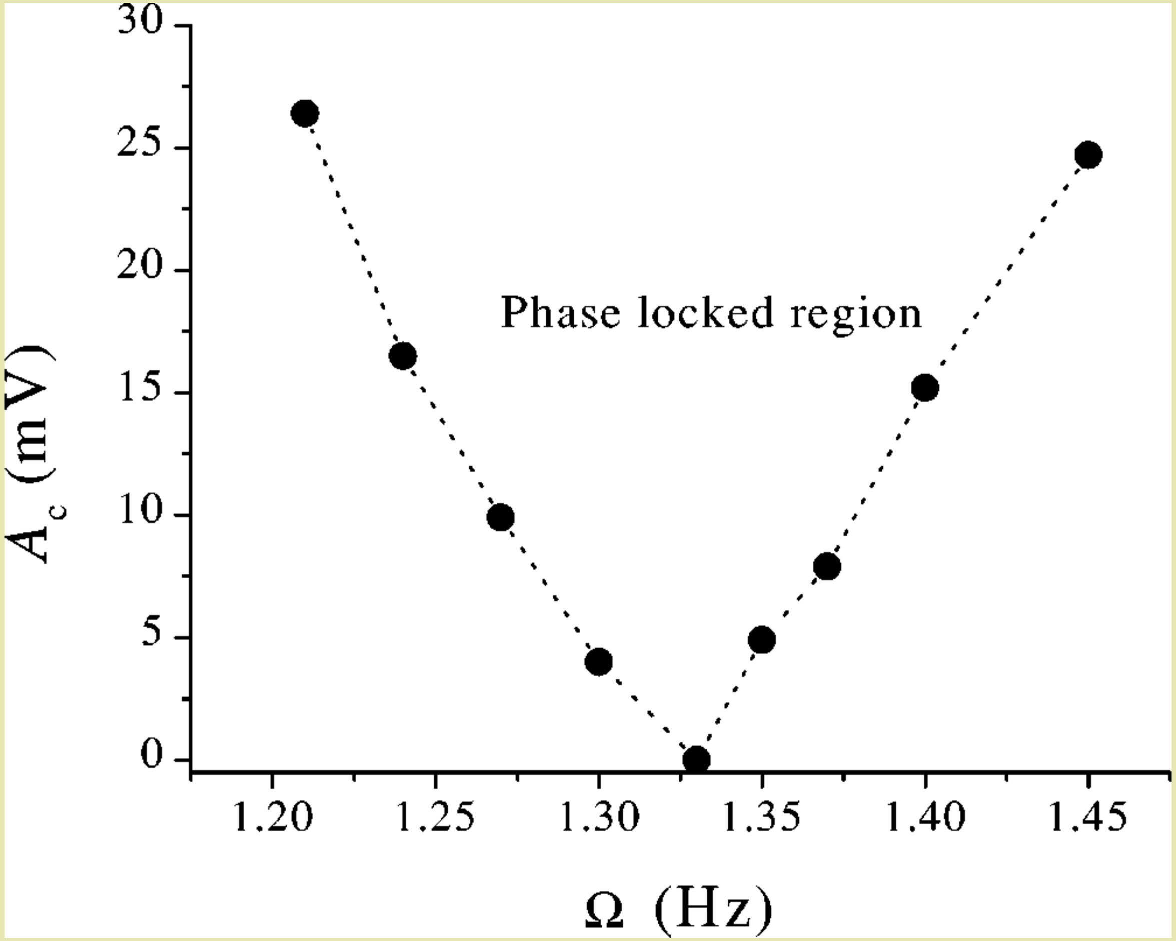


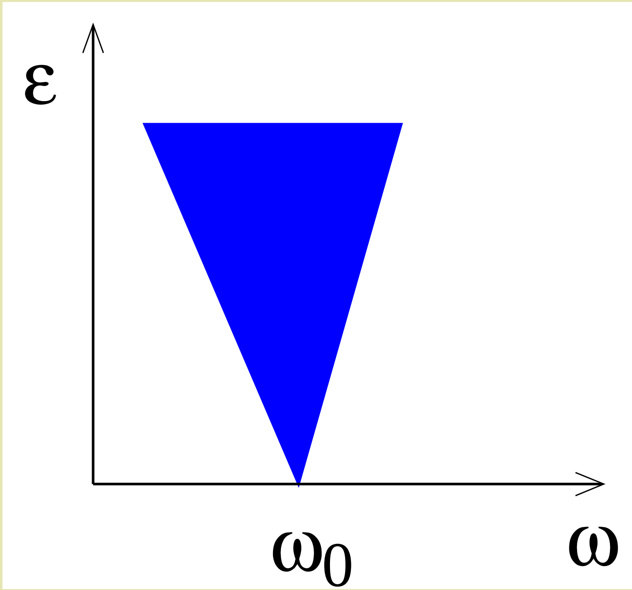
Frequency difference as a function of driving frequency for different amplitudes of forcing:



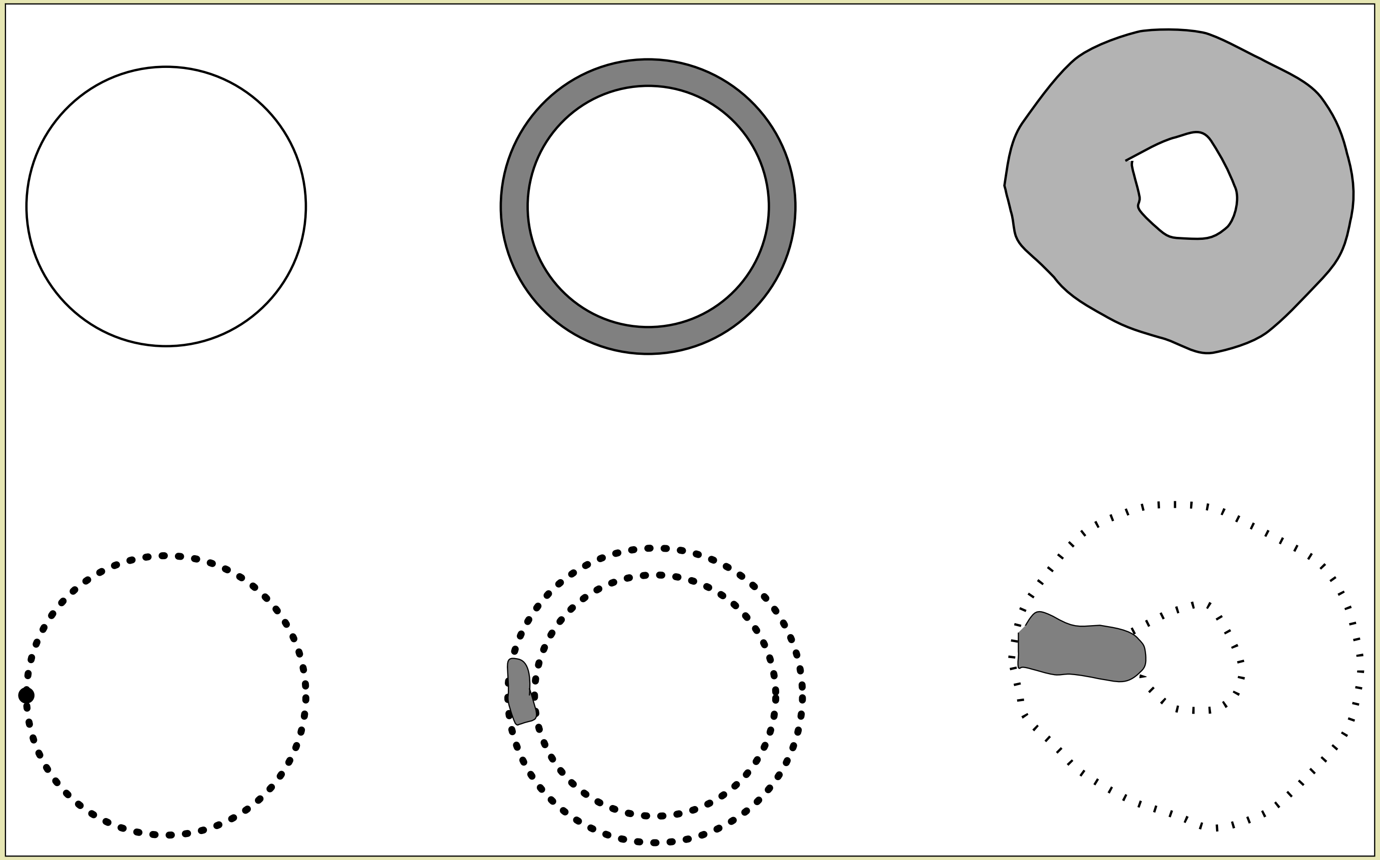


Synchronization region:





Unified description of regular, noisy, and chaotic oscillators



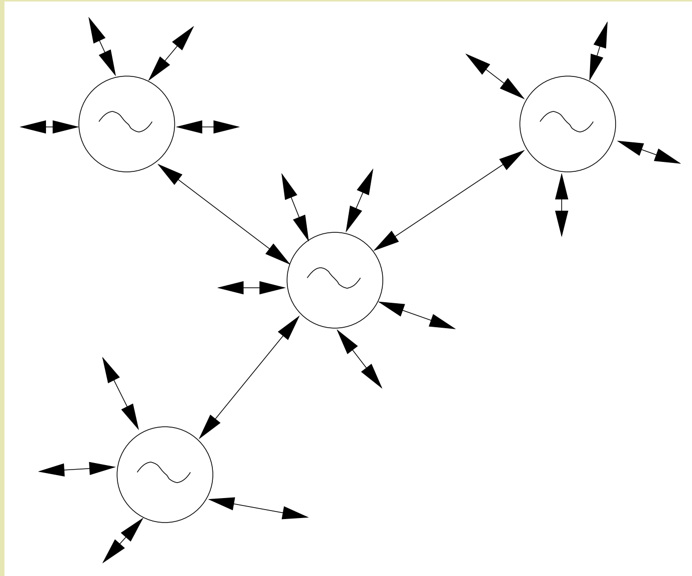
autonomous oscillators

forced oscillators

periodic **noisy** **chaotic**

Ensembles of globally (all-to-all) couples oscillators

- Physics: arrays of Josephson junctions, multimode lasers,...
- Biology and neuroscience: cardiac pacemaker cells, population of fireflies, neuronal ensembles,...
- Social behavior: applause in a large audience, dance,...



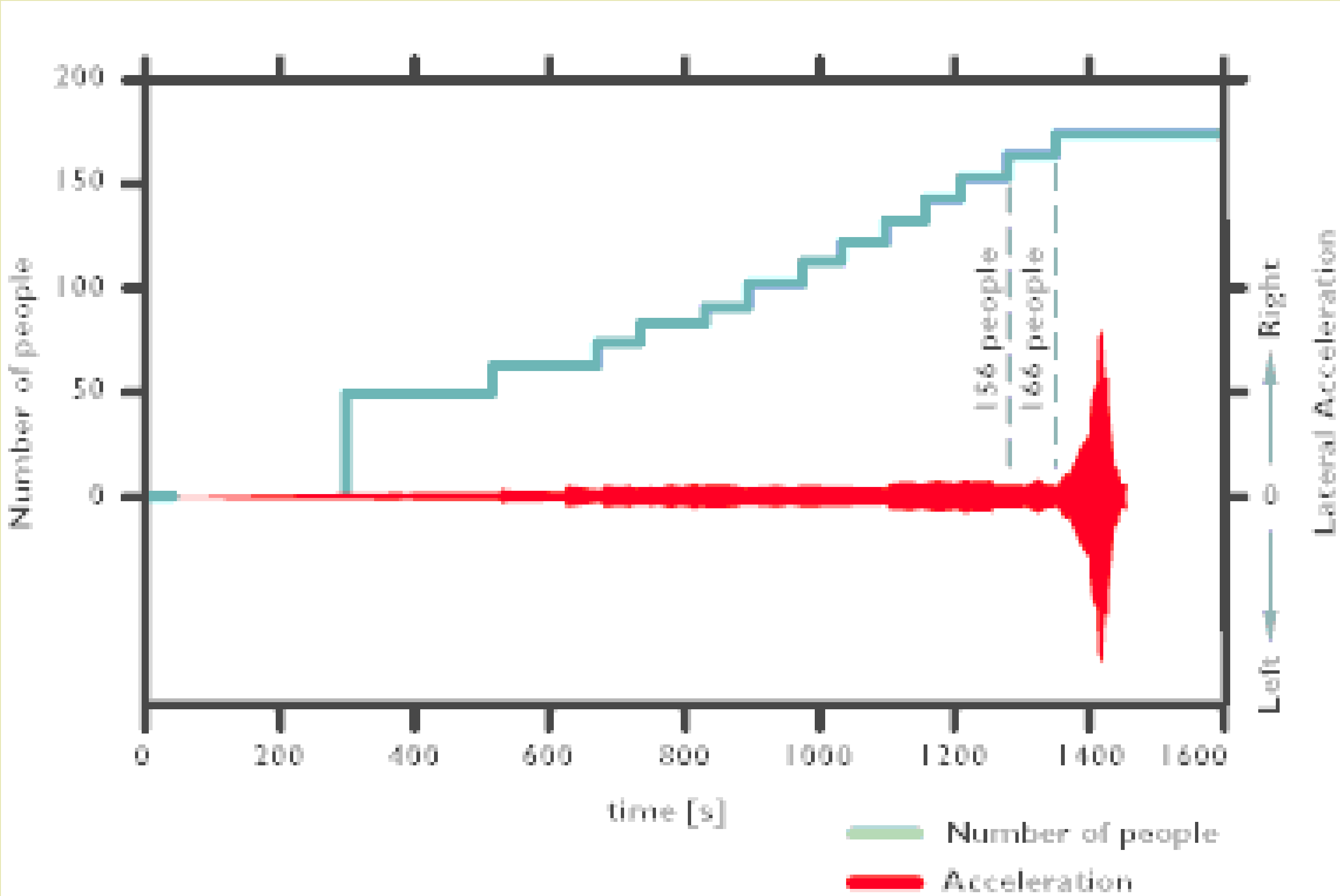
Mutual coupling adjusts phases of indvidual systems, which start to keep pace with each other

Synchronization appears as

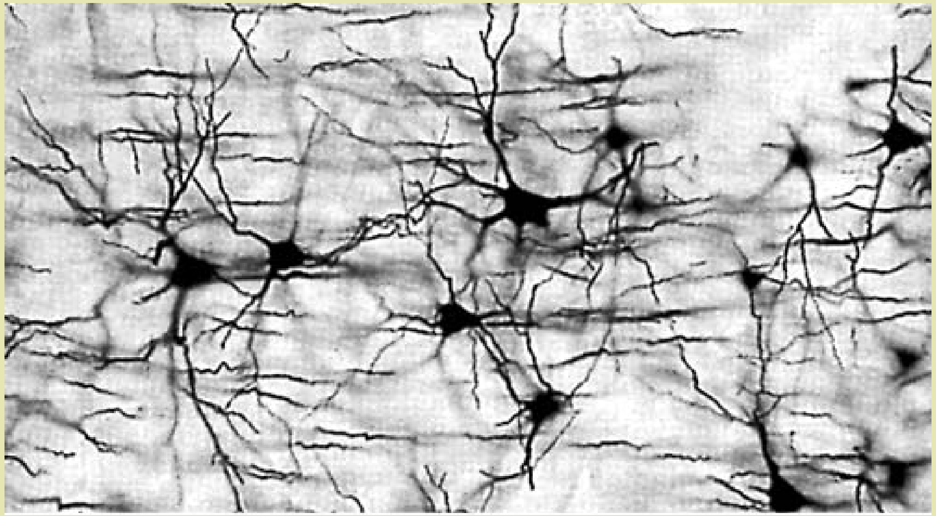
a nonequilibrium order-disorder transition

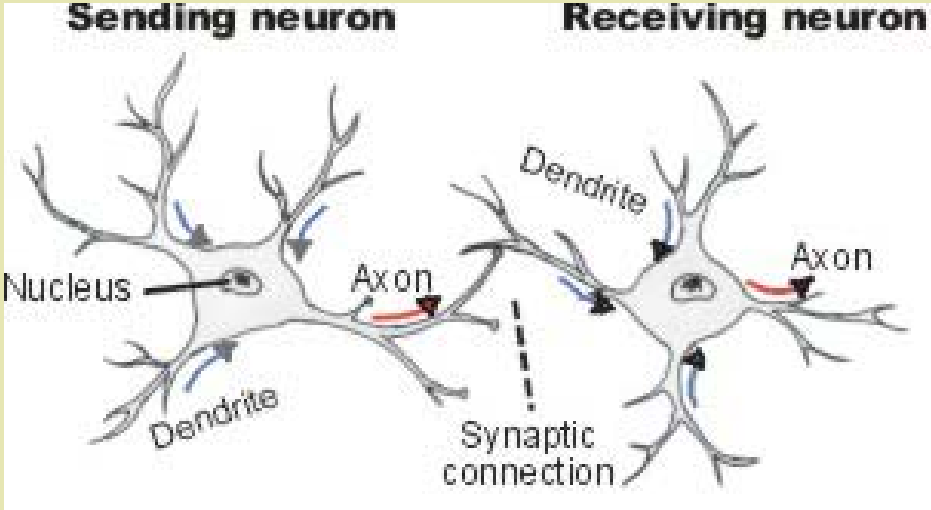
A macroscopic example: Millenium Bridge

Experiment with Millenium Bridge



Coupled neurons





Synchronisation in neuronal ensembles is believed to be the reason for emergence of pathological rhythms in the **Parkinson disease** and in the **Epilepsy**

Watching synchrony

Watching sinchronous blinking of fireflies has boomed into an industry at Kuala Selangor firefly park (Malaysia), see www.fireflypark.com



Kuramoto model: coupled phase oscillators

Phase oscillators with all-to-all coupling (like Adler equation)

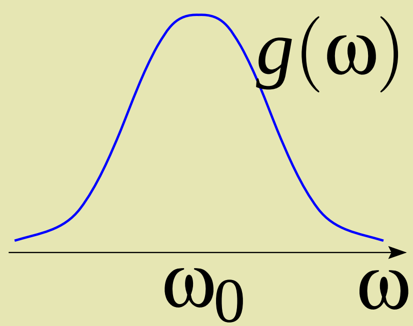
$$\dot{\phi}_k = \omega_k + \varepsilon \frac{1}{N} \sum_{j=1}^{N} \sin(\phi_j - \phi_k) = \omega_k + \varepsilon K \sin(\Theta - \phi_k)$$

System can be written as a mean-field coupling with the mean field (complex order parameter)

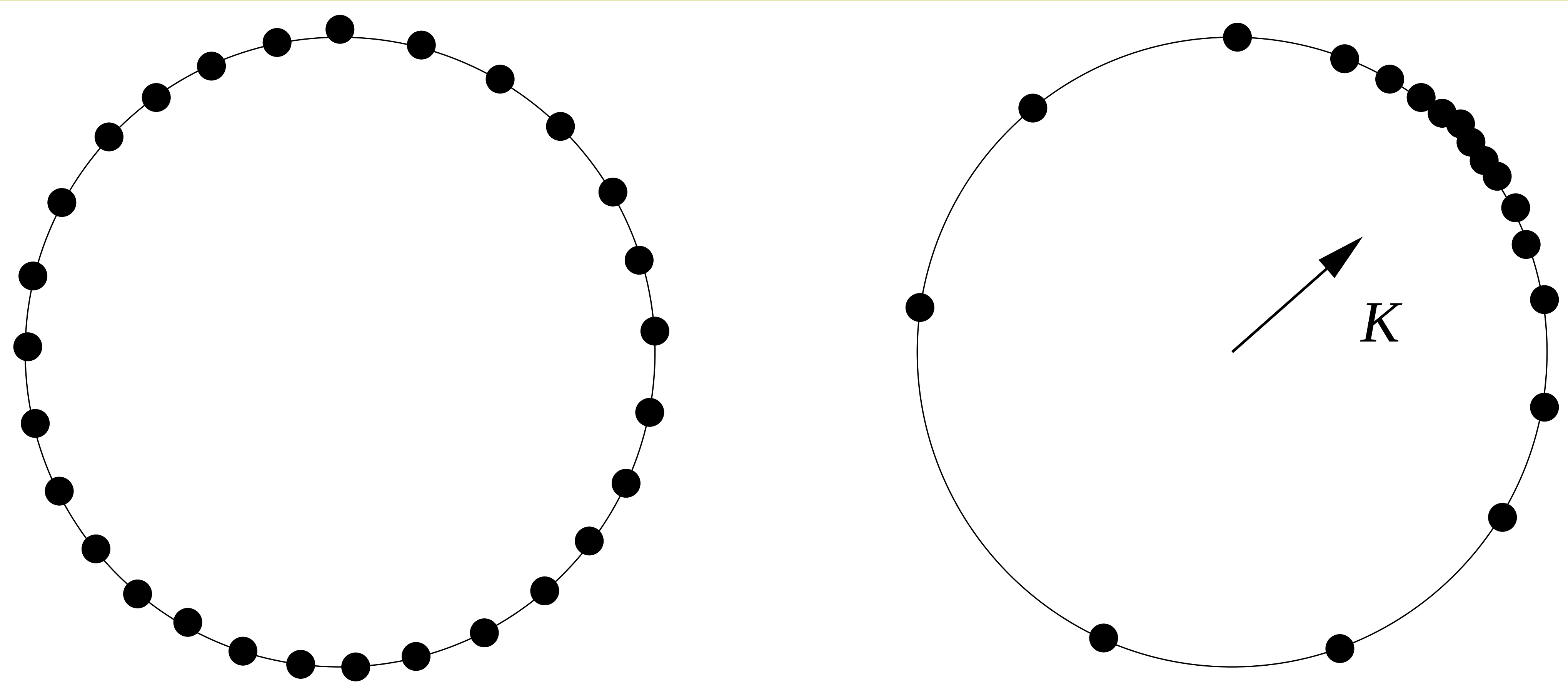
$$K e^{i\Theta} = \frac{1}{N} \sum_k e^{i\phi_k}$$

The natural frequencies are distributed around some mean frequency ω_0

$\approx$ finite temperature



Synchronisation transition



small ε: no synchronization, phases are distributed uniformly, mean field $= 0$

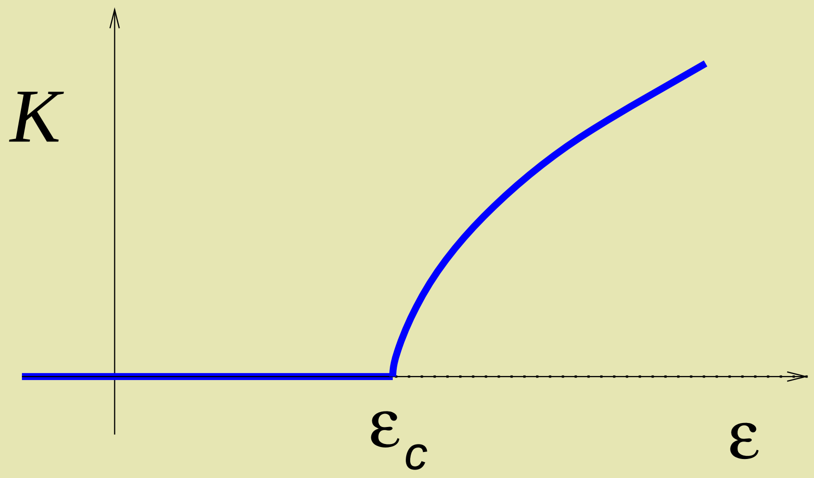
large ε: synchronization, distribution of phases is non-uniform, mean field $\neq 0$

Theory of transition

Similar to the mean-field theory of ferromagnetic transition:

a self-consistent equation for the mean field

$$Ke^{i\Theta} = \int_0^{2\pi} n(\phi) e^{i\phi}\, d\phi = K\varepsilon \int_{-\pi/2}^{\pi/2} g(K\varepsilon \sin\phi) \cos\phi \, e^{i\phi}\, d\phi$$



Critical coupling ε_c $\sim$ width of distribution $g(\omega) \sim$ “temperature”

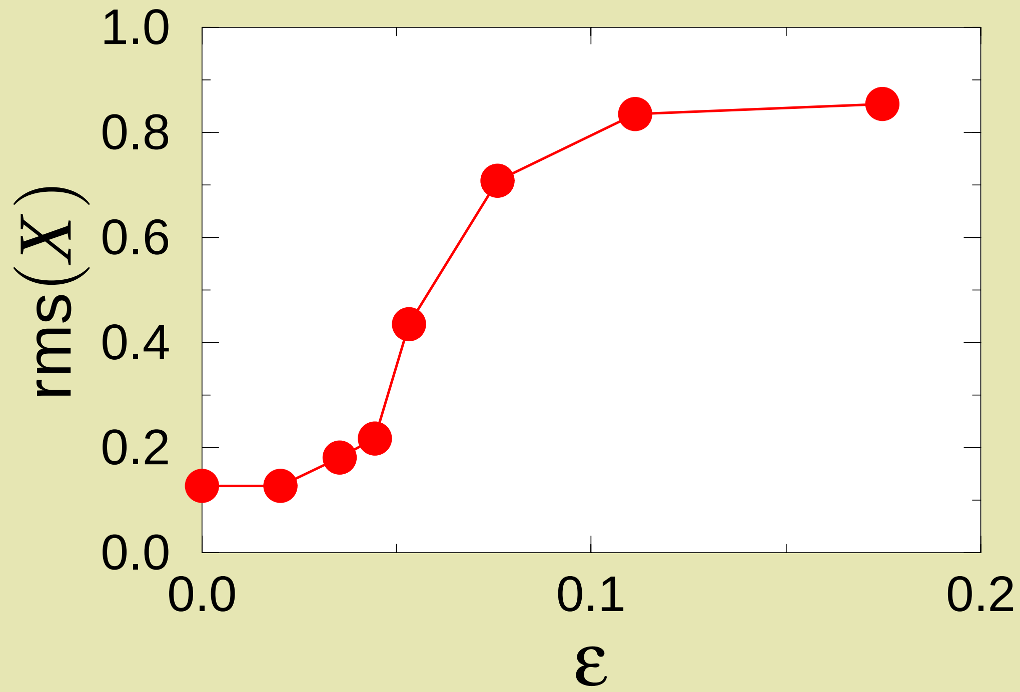
Experiment

Experimental example: synchronization transition

in ensemble of 64 chaotic electrochemical oscillators

Kiss, Zhai, and Hudson, Science, 2002

Finite size of the ensemble yields fluctuations of the mean field $\sim \frac{1}{N}$



Identical oscillators: zero temperature

All frequencies are equal, $\varepsilon > 0$, additional phase shift β in coupling

$$\dot{\phi}_k = \omega + \varepsilon \frac{1}{N} \sum_{j=1}^{N} \sin(\phi_j - \phi_k - \beta) = \omega + \varepsilon K \sin(\Theta - \phi_k - \beta)$$

Attraction: $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$ $\Longrightarrow$

Synchronization, all phases identical $\phi_1 = \ldots = \phi_N = \Theta$,

maximal order parameter $K = 1$

Repulsion: $-\pi < \beta < -\frac{\pi}{2}$ and $\frac{\pi}{2} < \beta < \pi$ $\Longrightarrow$

Asynchrony, phases distributed uniformely,

order parameter vanishes $K = 0$

Linear vs nonlinear coupling I

- Synchronization of a periodic autonomous oscillator is a nonlinear phenomenon
- it occurs already for infinitely small forcing
- because the unperturbed system is singular (zero Lyapunov exponent)

In the Kuramoto model "linearity" with respect to forcing is assumed

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) + \varepsilon_1 \mathbf{f}_1(t) + \varepsilon_2 \mathbf{f}_2(t) + \cdots$$

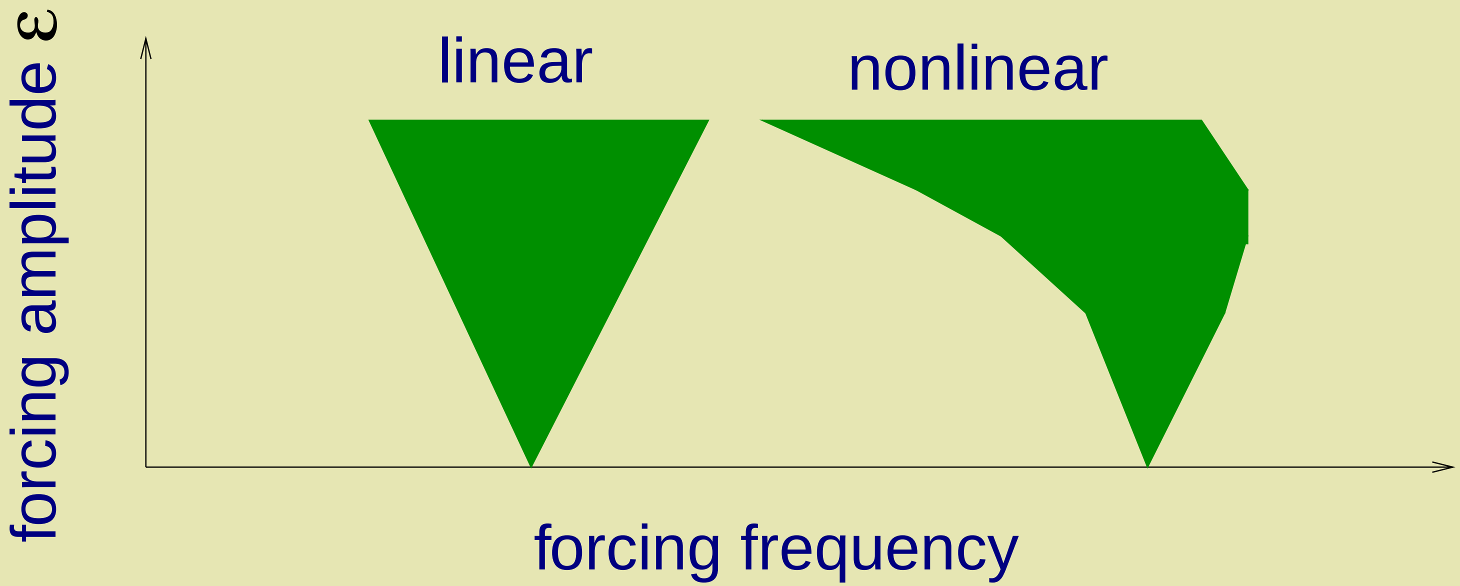
$$\dot{\phi} = \omega + \varepsilon_1 q_1(\phi, t) + \varepsilon_2 q_2(\phi, t) + \cdots$$

Linear vs nonlinear coupling II

Strong forcing leads to “nonlinear” dependence on the forcing amplitude

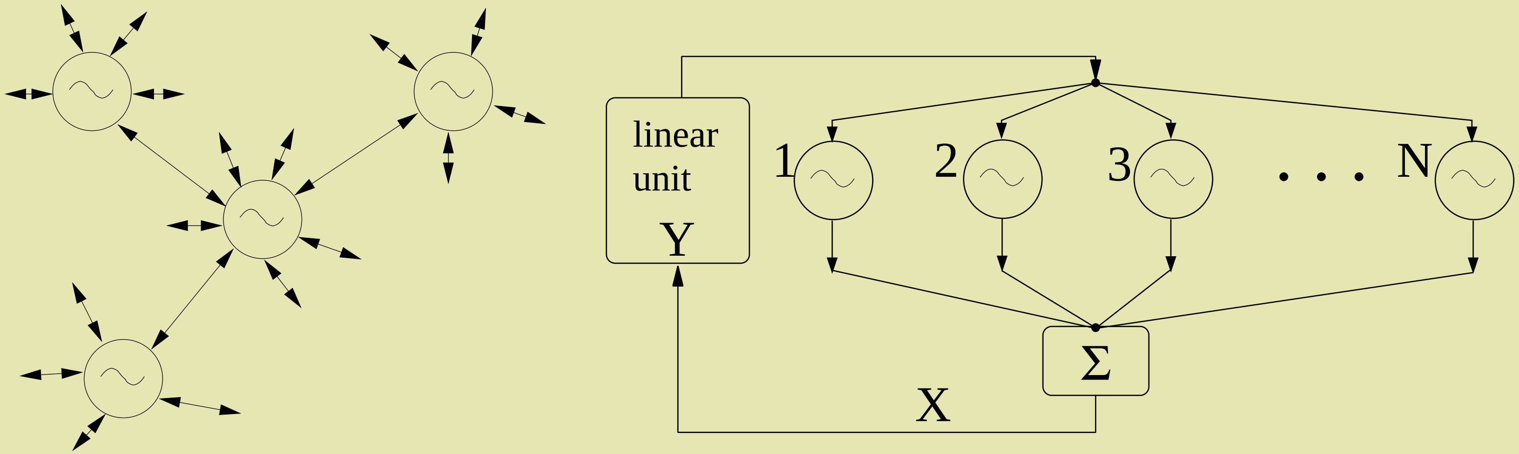
$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) + \varepsilon \mathbf{f}(t)$$
$$\dot{\phi} = \omega + \varepsilon q^{(1)}(\phi, t) + \varepsilon^2 q^{(2)}(\phi, t) + \cdots$$

Nonlineraity of forcing manifests itself in the deformation/skeweness of the Arnold tongue and in the amplitude depnedence of the phase shift

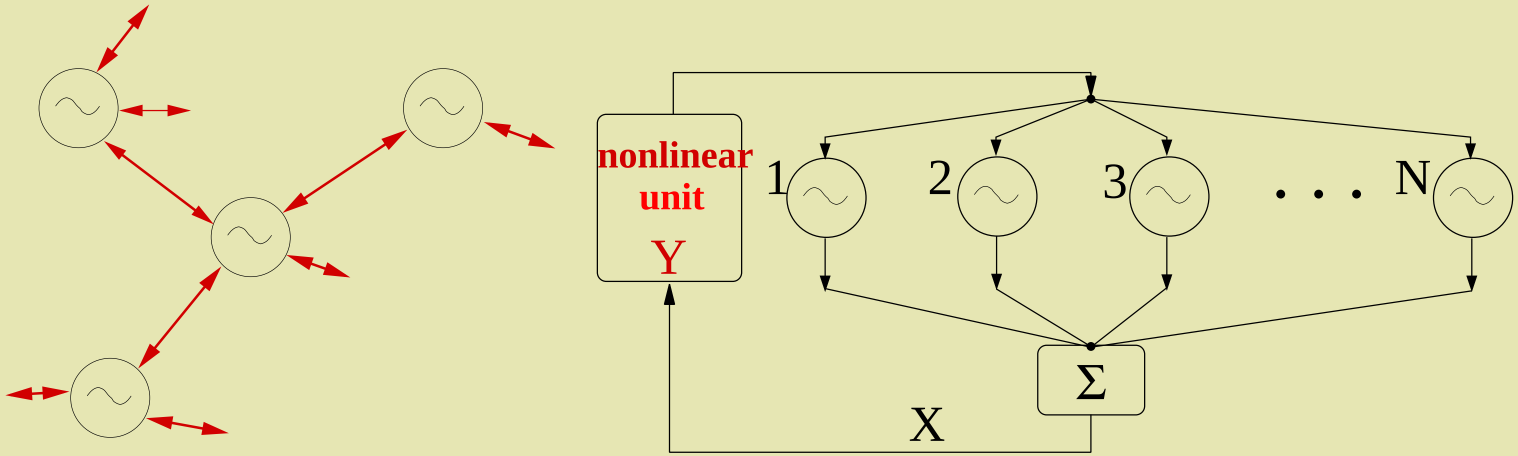


Linear vs nonlinear coupling III

Small each-to-each coupling $\Longleftrightarrow$ coupling via linear mean field



Strong each-to-each coupling $\Longleftrightarrow$ coupling via nonlinear mean field



Nonlinear coupling: a minimal solvable model

We take the Kuramoto model and assume

nonlinear dependence of **coupling strength** R and **phase shift** β

on the order parameter K

$$\boxed{\dot{\phi}_k = \omega + R(\varepsilon K)\varepsilon K \sin(\Theta - \phi_k + \beta(\varepsilon K)) \qquad K e^{i\Theta} = \frac{1}{N}\sum_k e^{i\phi_k}}$$

For $R = const$ and $\beta = const$ the Kuramoto model at zero temperature is restored

Desynchronization transition: critical coupling

$$\boxed{\dot{\phi}_k = \omega + R(\varepsilon K)\varepsilon K \sin(\Theta - \phi_k + \beta(\varepsilon K))}$$

Synchronous solution $\phi_k = \Theta$ and $K = 1$

is stable if $-\pi/2 < \beta(1,\varepsilon) < \pi/2$

$\Longrightarrow$ critical coupling is determined by $\boxed{\beta(1,\varepsilon_q) = \pm\pi/2}$

Transition from attraction to repulsion at a critical coupling strength

Beyond this transition **partial synchronization**

with $0 < K < 1$ is observed

The mean field has frequency $\Omega = \dot{\Theta} \neq \omega_{osc} = \langle\dot{\phi}\rangle$

This field does not entrain the oscillators

$\Longrightarrow$ **quasiperiodic regimes** are observed

Equations for the phase difference

$$\frac{d\phi_k}{dt} = \omega + R(K,\varepsilon)K\sin(\Theta - \phi_k + \beta(K,\varepsilon))$$

$$\frac{d\Theta}{dt} = \Omega$$

we obtain for $\psi_k = \phi_k - \Theta$

$$\frac{d\psi_k}{dt} = \omega - \Omega + R(K,\varepsilon)K\sin(\beta(K,\varepsilon) - \psi_k)$$

- following Kuramoto: we consider $N \longrightarrow \infty$ and drop the indices
- from the definition $Ke^{i\Theta} = \langle e^{i\phi} \rangle$

$\Longrightarrow$ **self-consistency condition** $K = \langle e^{i\psi} \rangle = \int_{-\pi}^{\pi} e^{i\psi}\rho(\psi)d\psi$

Self-consistency condition

- complex equation $K = \int_{-\pi}^{\pi} e^{i\psi}\rho(\psi)d\psi$ for determination of K and Ω
- probability distribution $\rho(\psi) \sim |\dot{\psi}|^{-1}$ can be explicitely obtained after normalization
- self-consistent equation yelds

$$\beta(K,\varepsilon) = \pm\pi/2 \qquad \Omega = \omega \pm R(K,\varepsilon)(1+K^2)/2$$

- with account of this, integration of equation for $\dot{\psi}$ yelds

$$\omega_{osc} = \omega \pm RK^2$$

Self-organized quasiperiodicity

- frequencies Ω and ω_{osc} depend on ε in a smooth way

  $\Longrightarrow$ generally we observe a quasiperiodicity

- recall the equation for critical coupling $\boxed{\beta(1,\varepsilon_q) = \pm\pi/2}$

  and compare with just obtained $\boxed{\beta(K,\varepsilon) = \pm\pi/2}$

- attracting coupling for small mean field

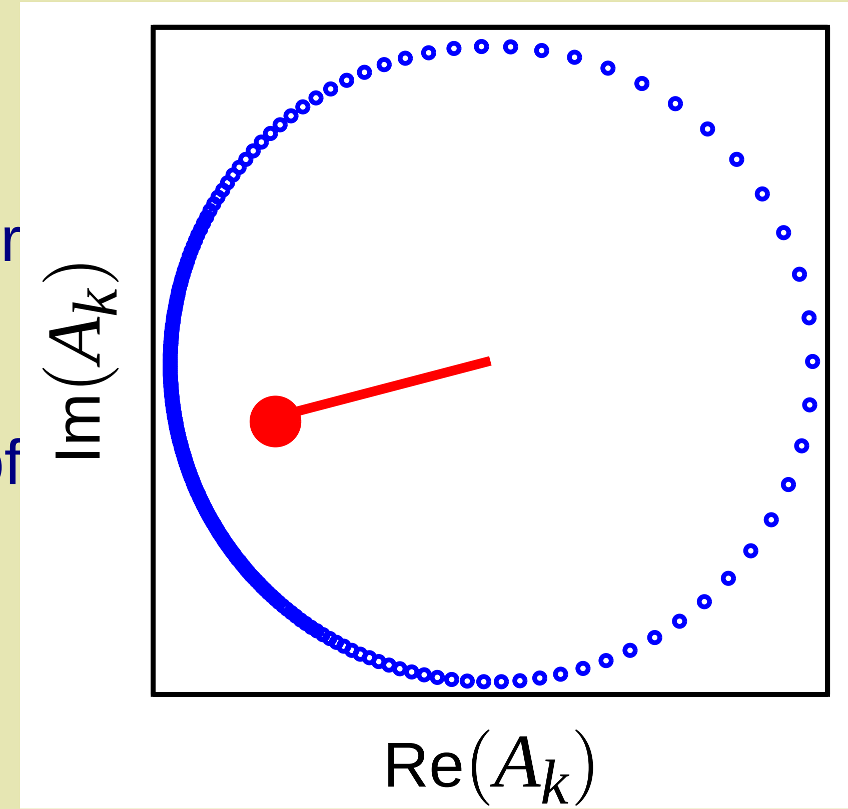
  repulsing coupling for large mean field

  $\Longrightarrow$ the system sets on exactly on the stabilty border, i.e. in a

  self-organized critical state

Simulation

- non-uniform distribution of oscillator phases, here for $\varepsilon - \varepsilon_q = 0.2$
- different velocities of oscillators and of the mean field

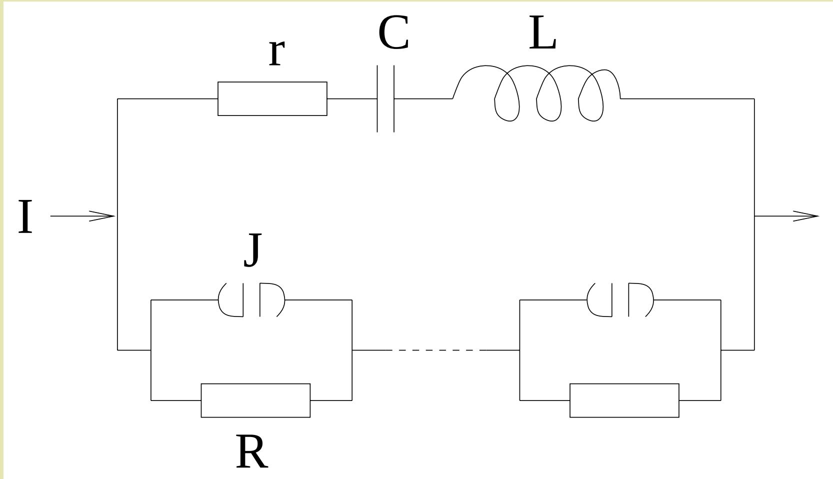


Example: Josephson junctions

- array of Josephson junctions, shunted by a common RLC-load (capacitances are neglected)

$$\frac{\hbar}{2eR}\frac{d\phi_k}{dt}+I_c\sin\phi_k=I-\frac{dQ}{dt}$$

$$\frac{d\Phi}{dt}+r\frac{dQ}{dt}+\frac{Q}{C}=\frac{\hbar}{2e}\sum_k\frac{d\phi_k}{dt}$$

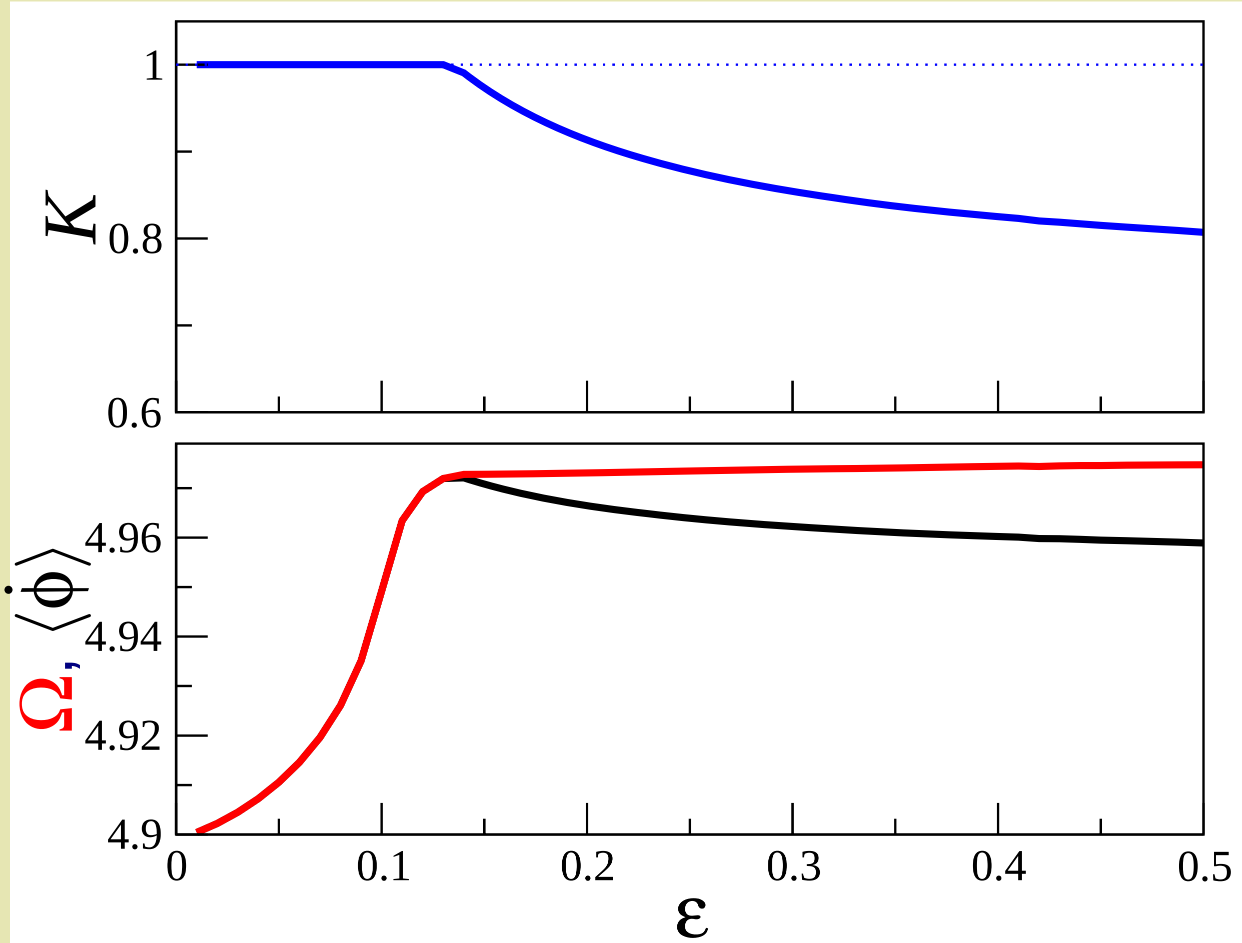


- the case of linear RLC-load can be reduced to the Kuramoto model (Wiesenfeld & Swift, 1995)
- we consider **nonlinear** load: the magnetic flux $\Phi=L_0\dot{Q}+L_1\dot{Q}^3$
- phase equation is of general nonlinear coupling type

Josephson junctions: numerical results

Critical coupling

$$\varepsilon_q \approx 0.13$$



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